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ERROR ESTIMATES FOR THE TIME DISCRETIZATION FOR NONLINEAR MAXWELL'S EQUATIONS*

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Abstract

This paper is devoted to the study of a nonlinear evolution eddy current model of the type $\partial_t \boldsymbol{B}(\boldsymbol{H}) + \nabla \times (\nabla \times \boldsymbol{H}) = \boldsymbol{0}$ subject to homogeneous Dirichlet boundary conditions $\boldsymbol{H} \times \boldsymbol{\nu} = \boldsymbol{0}$ and a given initial datum. Here, the magnetic properties of a soft ferromagnet are linked by a nonlinear material law described by $\boldsymbol{B}(\boldsymbol{H})$. We apply the backward Euler method for the time discretization and we derive the error estimates in suitable function spaces. The results depend on the nonlinearity of $\boldsymbol{B}(\boldsymbol{H})$.

Mathematics subject classification: 65M15, 83C50.
Key words: Electromagnetic field, Nonlinear eddy current problem, Time discretization, Error estimate.

1. Introduction

Eddy current problems are described by quasistationary Maxwell's equations

$$\begin{aligned} \nabla \times \boldsymbol{H} &= \boldsymbol{J}, \\ \partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} &= \boldsymbol{0}, \end{aligned} \tag{1.1}$$

where $\boldsymbol{H}$ denotes the magnetic field, $\boldsymbol{J}$ is the current density (current per unit area), $\boldsymbol{B}$ stands for the magnetic induction and $\boldsymbol{E}$ is the electric field. The quasistationary Maxwell equations can be obtained from the full Maxwell system omitting the term $\varepsilon_0 \partial_t \boldsymbol{E}$, where ε_0 is the permittivity of the free space, which has some strong physical interpretations.

The time dependent magnetic variables are related as follows

$$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}) = \boldsymbol{B}(\boldsymbol{H}), \tag{1.2}$$

where μ_0 denotes the magnetic permeability of free space and $\boldsymbol{M}$ describes the magnetization. The relation between $\boldsymbol{B}$ and $\boldsymbol{H}$ is nonlinear in ferromagnetic materials. Neglecting the hysteresis effects, the relationship $\boldsymbol{B}(\boldsymbol{H})$ is strictly monotone and invertible. The usual form is $\boldsymbol{B} = b(|\boldsymbol{H}|)\boldsymbol{H}$ or $\boldsymbol{H} = \nu(|\boldsymbol{B}|)\boldsymbol{B}$.

Taking into account Ohm's law

$$\boldsymbol{J} = \sigma \boldsymbol{E},$$

where σ is the conductivity (which can be a tensor in anisotropic materials), and eliminating the electric field $\boldsymbol{E}$ we arrive at

$$\partial_t \boldsymbol{B}(\boldsymbol{H}) + \nabla \times \left(\sigma^{-1} \nabla \times \boldsymbol{H}\right) = \boldsymbol{0}. \tag{1.3}$$

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Another equivalent form expressed in terms of the vector potential $\boldsymbol{u}$ for the magnetic field density $\boldsymbol{B} = \nabla \times \boldsymbol{u}$ reads as

$$\sigma \partial_t \boldsymbol{u} + \nabla \times (\nu(|\nabla \times \boldsymbol{u}|)\nabla \times \boldsymbol{u}) = \mathbf{0}. \tag{1.4}$$

The formulation (1.4) along with the homogeneous Dirichlet boundary condition has been analyzed in [1] under the following assumptions for the continuous function $\nu : \mathbb{R}_0^+ \to \mathbb{R}^+$

$$0 < \nu_1 \le \nu(s) \le \nu_2 \qquad \forall s \in \mathbb{R}_0^+$$

and $\nu(s)s$ is strictly monotone and Lipschitz continuous. The authors addressed the solvability of the problem and they proposed a numerical treatment by the so-called multiharmonic approach.

The nonlinear PDE of the type (1.3) or (1.4) have some applications in superconductors – see [2,3]. It is well known that high-field (hard) type-II superconductors are not ideal conductors of electric current. They can be treated as electrically nonlinear conductors. The process of electromagnetic field penetration in such devices is the process of nonlinear diffusion. The equation describing the process can degenerate. For an overview of models with some hierarchy structure we refer the reader to [4, 5]. The magnetization of type-II superconductors in a nonstationary external magnetic field can also be formulated in terms of a scalar $p-$Laplacian equation if the magnetic field lies only in one direction. This situation has been studied in many papers. Authors in [6] showed that the limit as $p \to \infty$ for the scalar p-Laplacian is a solution to Bean's model. The Hölder continuity of solution has been analyzed in [7]. The 2-dimensional problem was studied in [8] using the theory of nonlinear semigroups.

Slodička in [9] applied the backward Euler scheme to (1.3) for the discretization in time and he derived the error estimates for a degenerate problem. A similar technique was used in [10] for an application in superconductors. The error estimates for the time-discretization in both papers were $\mathcal{O}(\tau^{\frac{1}{2}})$ – thus suboptimal.

Paper [11] was devoted to the fix-point approximation of a nonlinear steady-state problem, which arises from the backward Euler discretization of (1.3) at each time step. The authors proved the convergence of iterations for both, the Lipschitz continuous and the degenerate cases.

The main goal of this study is to derive the error estimates for the backward Euler scheme applied to (1.3) along with the homogeneous Dirichlet boundary condition and a given initial datum. We distinguish between the Lipschitz continuous nonlinearity and the degenerate case. Our results are formulated in Theorems 3.1-3.3. The convergence rate is optimal for the Lipschitz nonlinearity, i.e. $\mathcal{O}(\tau)$ – see Theorem 3.1. In the degenerate case we improved the error estimate from [9,10] for some type of nonlinearity – see Theorem 3.3. The last section is devoted to the numerical experiments to support our theoretical results.

2. Preliminaries

Without loss of generality we assume that $\mu_0 = \sigma = 1$. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a Lipschitz continuous boundary $\Gamma = \partial\Omega$. We denote by $(\boldsymbol{w}, \boldsymbol{z})$ be the usual L_2-inner product of any real or vector-valued functions $\boldsymbol{w}$ and $\boldsymbol{z}$ in Ω, i.e.,

$$(\boldsymbol{w}, \boldsymbol{z}) = \int_\Omega \boldsymbol{w} \cdot \boldsymbol{z}, \quad \|\boldsymbol{w}\| = \sqrt{(\boldsymbol{w}, \boldsymbol{w})}.$$