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A FEM-BEM FORMULATION FOR AN EXTERIOR QUASILINEAR ELLIPTIC PROBLEM IN THE PLANE*

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Dedicated to Professor Junzhi Cui on the occasion of his 70th birthday

Abstract

In this paper, the finite element method and the boundary element method are combined to solve numerically an exterior quasilinear elliptic problem. Based on an appropriate transformation and the Fourier series expansion, the exact quasilinear artificial boundary conditions and a series of the corresponding approximations for the given problem are presented. Then the original problem is reduced into an equivalent problem defined in a bounded computational domain. We provide error estimate for the Galerkin method. Numerical results are presented to illustrate the theoretical results.

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1. Introduction

In this paper, we consider a discretization procedure for an exterior quasilinear problem which combines the finite element method (FEM) and the boundary element method (BEM). This technique has been used to solve many linear problems, see, e.g., [6, 10, 15–17]. It has also been successfully generalized to nonlinear boundary value problems [4, 8, 9, 12]. In these extensions, the error analysis is often given when the coefficients satisfy conditions that make the nonlinear operator strongly monotone and Lipschitz continuous, see, e.g., [4, 9]. The advantage in this case is that Céa's lemma is satisfied. When these conditions do not hold, Xu [13] provides a useful tool by linearizing the nonlinear partial differential equation at a given isolated solution and considering its finite element discretization. Meddahi [11] extends this approach and gives the error analysis. However, all the problems considered are subject to the assumptions that they are homogeneous and linear with constant coefficients outside a bounded domain. In this paper, we shall consider more general quasilinear problems on the exterior region and give the error estimates.

Let Ω_0 is a bounded and simple connected domain in $\mathbb{R}^2$ with sufficiently smooth boundary Γ_0. $\Omega := \mathbb{R}^2/\overline{\Omega_0}$. We consider continuous nonlinear functions α_{kl} and β_i: $\Omega \times \mathbb{R} \to \mathbb{R}$ $(i = 0, 1, 2; k, l = 1, 2)$ such that the derivatives $(\partial\beta_i/\partial s)$, $(\partial\alpha_{kl}/\partial s)$, $(\partial^2\beta_i/\partial s^2)$, $(\partial^2\alpha_{kl}/\partial s^2)$, $(\partial\beta_i/\partial x_j), (\partial\alpha_{kl}/\partial x_j) (j = 1, 2)$ are continuous in $\Omega \times \mathbb{R}$. We need to approximate a function u that satisfies

$$\begin{cases} -div(\alpha(x,u)\nabla u + \beta(x,u)) + \beta_0(x,u) = f(x) & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \Gamma_0, \\ u(x) = \mathcal{O}(1), & \text{when} \quad |x| \to +\infty, \end{cases} \tag{1.1}$$

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where $\beta(x,u) = (\beta_1(x,u), \beta_2(x,u))^T$, $\alpha(x,u) = (\alpha_{kl})_{k,l=1}^2$.

Some existence and uniqueness results for this type of problem are given in [5] under some conditions on the coefficients α, β_i. We will not consider such issues, but instead, we assume that (1.1) has at least one solution. Our main purpose is to provide artificial boundary conditions for general quasilinear problem and error estimates for an approximate solution obtained from a FEM-BEM discretization scheme.

Assume that the given function $\beta_0, \beta \in L^2(\Omega)$ and $f(x) \in L^2(\Omega)$ has compact support, i.e., there is a constant $R_0 > 0$, such that

$$supp\ \beta_0, supp\ \beta \subset \Omega_{R_0} := \{x \in \mathbb{R}^2 | |x| \le R_0\}, \qquad supp\ f(x) \subset \Omega_{R_0}.$$

Moreover, we assume that there exists constant $C_0 > 0$, such that

$$\xi^T \alpha(x,u)\xi \ge C_0|\xi|^2, \qquad \forall u \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^2, \quad x \in \overline{\Omega}_{R_0} \tag{1.2}$$

$$\alpha(x,u) = \tilde{\alpha}(u), \qquad \text{when} \quad |x| \ge R_0 \tag{1.3}$$

We introduce an artificial boundary

$$\Gamma_R = \{x \in \mathbb{R}^2 | |x| = R\} \quad \text{with} \quad R \ge R_0.$$

Γ_R divides Ω into two regions, a bounded domain $\Omega_i = \{x \in \Omega | |x| \le R\}$, and Ω_e which is the unbounded region exterior to Γ_R. Then the problem (1.1) can be rewritten in the coupled form:

$$\begin{cases} -div(\alpha(x,u)\nabla u + \beta(x,u)) + \beta_0(x,u) = f(x) & \text{in} \quad \Omega_i, \\ u = 0, & \text{on} \quad \Gamma_0, \end{cases} \tag{1.4}$$

$$\begin{cases} -div(\tilde{\alpha}(u)\nabla u) = 0 & \text{in} \quad \Omega_e, \\ u(x) = \mathcal{O}(1), & \text{when} \ |x| \to +\infty, \end{cases} \tag{1.5}$$

$$u(x) \text{ and } \tilde{\alpha}(u)\partial u/\partial n \text{ are continuous on } \Gamma_R. \tag{1.6}$$

Obviously, if $\alpha(x,u) \equiv a$ when $|x| \ge R_0$, the problem (1.5) is simplified to a linear exterior elliptic problem [11].

We introduce the so-called Kirchhoff transformation

$$w(x) = \int_0^{u(x)} \tilde{\alpha}(\xi)d\xi, \qquad x \in \Omega_e \tag{1.7}$$

which gives

$$\nabla w = \tilde{\alpha}(u)\nabla u. \tag{1.8}$$

From (1.5) we have that w satisfies the following problem

$$\begin{cases} -\triangle w = 0 & \text{in} \quad \Omega_e, \\ w(x) = \mathcal{O}(1), & \text{when} \ |x| \to +\infty. \end{cases} \tag{1.9}$$

Let W_p^m be the standard Sobolev spaces with norm $\|\cdot\|_{m,p,\Omega_i}$ and semi-norms $|\cdot|_{m,p,\Omega_i}$. For $p = 2$, we denote $H^m(\Omega_i) = W_2^m$, $\|\cdot\|_{m,\Omega_i} = \|\cdot\|_{m,2,\Omega_i}$ and $|\cdot|_{m,\Omega_i} = |\cdot|_{m,2,\Omega_i}$.

The rest of this paper is organized as follows. In Section 2, we give the exact quasilinear artificial boundary condition on the artificial boundary, and present a new version of FEM-BEM formulation. In Section 3, the error analysis of the coupling method is given. Finally, Section 4 is devoted to numerical experiments to illustrate our theoretical results.