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NUMERICAL BOUNDARY CONDITIONS FOR THE FAST SWEEPING HIGH ORDER WENO METHODS FOR SOLVING THE EIKONAL EQUATION*

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Dedicated to Professor Junzhi Cui on the occasion of his 70th birthday

Abstract

High order fast sweeping methods have been developed recently in the literature to solve static Hamilton-Jacobi equations efficiently. Comparing with the first order fast sweeping methods, the high order fast sweeping methods are more accurate, but they often require additional numerical boundary treatment for several grid points near the boundary because of the wider numerical stencil. It is particularly important to treat the points near the inflow boundary accurately, as the information would flow into the computational domain and would affect global accuracy. In the literature, the numerical solution at these boundary points are either fixed with the exact solution, which is not always feasible, or computed with a first order discretization, which could reduce the global accuracy. In this paper, we discuss two strategies to handle the inflow boundary conditions. One is based on the numerical solutions of a first order fast sweeping method with several different mesh sizes near the boundary and a Richardson extrapolation, the other is based on a Lax-Wendroff type procedure to repeatedly utilizing the PDE to write the normal spatial derivatives to the inflow boundary in terms of the tangential derivatives, thereby obtaining high order solution values at the grid points near the inflow boundary. We explore these two approaches using the fast sweeping high order WENO scheme in [18] for solving the static Eikonal equation as a representative example. Numerical examples are given to demonstrate the performance of these two approaches.

Mathematics subject classification: 65N06, 65N22.
Key words: Fast sweeping method, WENO scheme, Boundary condition.

1. Introduction

In this paper we are interested in the numerical solution of two dimensional static Hamilton-Jacobi equations

$$H(\phi_x, \phi_y) = f(x, y) \tag{1.1}$$

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which is defined on a domain Ω with suitable boundary conditions. Typically, the boundary condition for the solution ϕ is provided in the inflow part Γ of the boundary. In particular, we will only study the so-called Eikonal equation in this paper as an example, that is, the Hamiltonian H in (1.1) is given by

$$H(u,v) = \sqrt{u^2+v^2}. \tag{1.2}$$

Notice that the solution to (1.1) may not always be differentiable or unique, and we are interested in the viscosity solution [4] which is unique, Lipschitz continuous, but may not be everywhere differentiable.

Applications in which the Hamilton-Jacobi equation (1.1), in particular the Eikonal equation (1.1)-(1.2), appears are abundant, for example the level set method, image processing and computer vision, and control theory. Some of the recently developed pedestrian flow models [7, 16] also involve the static Eikonal equation.

Numerical discretization for (1.1) includes first order monotone schemes on structured meshes [5] and on unstructured meshes [1], high order essentially non-oscillatory (ENO) schemes on structured meshes [11, 12], high order weighted ENO (WENO) schemes on structured meshes [8], high order WENO schemes on unstructured meshes [17], and high order discontinuous Galerkin methods on unstructured meshes [3, 6], among many others. A review of the discretization techniques for the Hamilton-Jacobi equations can be found in [14].

For a time dependent Hamilton-Jacobi equation

$$\phi_t + H(\phi_x, \phi_y) = f(x,y), \tag{1.3}$$

an explicit time discretization, such as the total variation diminishing (TVD) time discretization in [15], is often used. Such discretization can also be used to obtain the steady state solution of (1.1), by marching in time until the difference of the numerical solution between successive time steps becomes negligibly small. This however may not be the most efficient approach to obtain the solution of (1.1). In recent years, the fast sweeping method has been developed as one of the efficient techniques for obtaining the steady state solution of (1.1). The original fast sweeping method [2, 19] is only for first order monotone schemes on structured meshes. For such first order schemes, there is no issue for numerical boundary conditions, since the first order upwind discretization will only need values from the physically given boundary condition on the inflow part of the domain boundary. Later, the fast sweeping method is generalized to some of the high order spatial discretizations. For example, in [18], the fast sweeping method is generalized to the high order WENO scheme of [8]; and in [10], it is generalized to the high order discontinuous Galerkin method of [3]. These high order fast sweeping methods are also used in the pedestrian flow simulations in [7, 16], which require repeated solution of a static Eikonal equation. The high order fast sweeping methods produce much more accurate solutions on coarser meshes when compared with the first order fast sweeping method. However, they do involve an additional difficulty associated with high order spatial discretizations, namely the necessity to treat numerical boundary conditions near the boundary. Our numerical experiments indicate that the main difficulty is near the inflow boundary, as simple extrapolation could take care of the outflow boundary since the information there would flow out of the computational domain. We will use the high order WENO scheme in [18] as a representative example to explain this difficulty. Because of the wider numerical stencil required for the high order WENO interpolation, the high order fast sweeping WENO method needs a suitable numerical boundary treatment for several grid points near the inflow boundary. In [18] and also several other papers