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A MONOTONE COMPACT IMPLICIT SCHEME FOR NONLINEAR REACTION-DIFFUSION EQUATIONS*

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Abstract

A monotone compact implicit finite difference scheme with fourth-order accuracy in space and second-order in time is proposed for solving nonlinear reaction-diffusion equations. An accelerated monotone iterative method for the resulting discrete problem is presented. The sequence of iteration converges monotonically to the unique solution of the discrete problem, and the convergence rate is either quadratic or nearly quadratic, depending on the property of the nonlinear reaction. The numerical results illustrate the high accuracy of the proposed scheme and the rapid convergence rate of the iteration.

Mathematics subject classification: 65M06, 65M12.
Key words: Nonlinear reaction-diffusion equation, Monotone compact implicit scheme, High accuracy, Monotone iteration, Rapid convergence rate.

1. Introduction

Many phenomena in physics, chemistry, biology and engineering are described by nonlinear reaction-diffusion equations. Much work has been done for the qualitative analysis of the equations (see [19] and references therein), as well as their numerical simulations (see, e.g., [7,10,13,17,18,20,21,23,24,28]). In this paper, we provide a new numerical treatment for a class of nonlinear reaction-diffusion equations. It includes the construction and analysis of a monotone compact implicit finite difference scheme with high accuracy, and an accelerated monotone iterative method with rapid convergence rate for solving the resulting discrete problem. The equation under consideration is of the form:

$$\begin{cases} \partial u/\partial t + \mathcal{L}u = f(x,t,u), & 0 < x < 1, \quad 0 < t \le T, \\ u(0,t) = g_0(t), \quad u(1,t) = g_1(t), & 0 < t \le T, \\ u(x,0) = u_0(x), & 0 \le x \le 1, \end{cases} \tag{1.1}$$

where $g_0(t)$, $g_1(t)$ and $u_0(x)$ are given continuous functions satisfying the compatibility conditions $u_0(0) = g_0(0)$ and $u_0(1) = g_1(0)$. The operator $\mathcal{L}u$ in (1.1) is given by

$$\mathcal{L}u = -\frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right), \tag{1.2}$$

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where the coefficient $k(x) \in C^1(0,1)$ and for certain constants α_0 and α_1,

$$0 < \alpha_0 \le k(x) \le \alpha_1, \qquad x \in (0,1). \tag{1.3}$$

The function $f(x,t,u)$ in (1.1) is continuous in its domain, and the function $f(\cdot,u)$, which is in general nonlinear in u, is continuously differentiable in u.

Various numerical methods have been developed for solving problem (1.1). In the usual finite difference methods, one approximates the term $\partial u/\partial t$ by Euler backward method and the differential operator $\mathcal{L}u$ by the central difference quotient (see, e.g., [7,10,13,17,18,20,21]). In this case, the resulting discrete system is tridiagonal, and so it does not need any fictitious points for implementing the scheme. However, such scheme has only the accuracy of $\mathcal{O}(\tau + h^2)$ where τ and h are the mesh sizes in time and in space, respectively (e.g., [15,17,18,20,21]). In other words, we must take small mesh sizes in order to obtain the desirable accuracy, and thus much computational work is involved.

As is well known, by using the Crank-Nicolson technique or the three-level Lees technique in the time discretization, the accuracy in time can be improved to second-order (see [4,15,25]). But if Lees technique is used, one has to evaluate the solution at the first time level by other method (see [4,15,25]). Another trick for improving accuracy in time is to use extrapolation technique (see [25]). For improvement of the accuracy in space, a conventional approach is to approximate $\mathcal{L}u$ by using more points in the space discretization (see [4]). However, this not only significantly increases the computational complexity but also causes difficulty in handling boundary conditions since fictitious points near boundaries are needed (see [4]).

An alternative approach of improving the accuracy in space is the so-called compact implicit method which has been developed and generalized by several investigators under the name *Operator Compact Implicit* (OCI) method (see, in particular, [2–4]). This method achieves the fourth-order accuracy while retaining the tridiagonal feature of a second-order method and not requiring additional fictitious points at the boundary (see [2–4,14]). Assume that the function $u(x)$ is independent of t. The main idea of the OCI method is to look for an approximation representation of $\mathcal{L}u$ by establishing the following relationship between $\mathcal{L}u$ and the function u on the three adjacent points of a uniform mesh $x_i = ih$ $(h = 1/L, i = 0, 1, \cdots, L)$:

$$r_i^- u_{i-1} + r_i^c u_i + r_i^+ u_{i+1} = q_i^-(\mathcal{L}u)_{i-1} + q_i^c(\mathcal{L}u)_i + q_i^+(\mathcal{L}u)_{i+1}, \qquad 1 \le i \le L-1, \tag{1.4}$$

or

$$\mathcal{R}_i u_i = \mathcal{Q}_i(\mathcal{L}u)_i, \qquad 1 \le i \le L-1,$$

where u_i and $(\mathcal{L}u)_i$ are the approximations to u and $\mathcal{L}u$ at x_i, respectively, and the operators $\mathcal{R}_i$ and $\mathcal{Q}_i$ are tridiagonal operators:

$$\mathcal{R}_i u_i = r_i^- u_{i-1} + r_i^c u_i + r_i^+ u_{i+1}, \quad \mathcal{Q}_i u_i = q_i^- u_{i-1} + q_i^c u_i + q_i^+ u_{i+1}, \qquad 1 \le i \le L-1. \tag{1.5}$$

This approximation representation for $\mathcal{L}u$ is *explicit* if $q_i^- = q_i^+ = 0$, and *implicit* otherwise. Without loss of generality, throughout this paper, (1.4) is assumed normalized so that

$$\lim_{h \to 0} q_i^c = \text{a positive constant}, \qquad 1 \le i \le L-1. \tag{1.6}$$

Following the terminology of [3,4], a scheme of the form (1.4) will be referred to as an *Operator Compact Implicit* (OCI) scheme if it is a fourth-order accurate approximation to $\mathcal{L}u$, i.e., if its truncation error is $\mathcal{O}(h^4)$ after normalization. Note that the fourth-order accuracy is the highest that can be obtained by a scheme of the form (1.4) (see, e.g., [3,14]).