Journal of Computational Mathematics, Vol.26, No.1, 2008, 69–75.

REAL ROOT ISOLATION OF SPLINE FUNCTIONS*

Renhong Wang and Jinming Wu
Institute of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China
Email: renhong@dlut.edu.cn, wujm97@yahoo.com.cn

Abstract

In this paper, we propose an algorithm for isolating real roots of a given univariate spline function, which is based on the use of Descartes' rule of signs and de Casteljau algorithm. Numerical examples illustrate the flexibility and effectiveness of the algorithm.

Mathematics subject classification: 65D07, 14Q05.
Key words: Real root isolation, Univariate spline, Descartes' rule of signs, de Casteljau algorithm.

1. Introduction

The relationship between the number of real roots of a univariate spline and the sequence of its B-spline coefficients has been studied by de Boor [1] and Goodman [5], which provides a new bounds on the number of real roots of the spline function. However, the specific distribution of real roots of a given univariate spline based on its signs and sizes of B-spline coefficients have not been investigated. The specific distribution can provide a good selection of initial approximations to all of its real roots in order to get started for iterative methods.

In 1989, Grandine [6] proposed a method for finding all real roots of a spline function based on the interval Newton method. It is primarily based on iteratively dividing the domain into segments that contain a zero, by using estimates for the derivatives of the spline function based on knot insertion. However, if we know the isolating intervals of a given spline function, then it will greatly reduce the computational cost for finding all of its real roots.

It is well known that there are several algorithms for polynomial real root isolation based on the use of Descartes' rule of signs, such as Uspensky's algorithm (see [2, 7] and references therein). It can be regarded as a preconditioned process for computing all the real roots of a given polynomial.

In this paper, we propose an algorithm for computing a sequence of disjoint intervals such that each of them contains exactly one real root of a given univariate spline, which is primarily based on the use of Descartes' rule of signs with its B-spline coefficients and de Casteljau algorithm. Numerical examples are also provided to illustrate the flexibility of the proposed algorithm.

2. Preliminaries

We begin by defining the class of spline functions of interest [8, 9]. Take integers $m, n \geq 0$ and a non-decreasing sequence $t = (t_0, t_1, \cdots, t_{m+n+1})$ with $t_i < t_{i+n+1},\ i = 0, 1, \cdots, m$. For

* Received October 11, 2006 / Revised version received April 4, 2007 / Accepted June 6, 2007 /

$i = 0, 1, \cdots, m$, let $N_{i,n}(x)$ denote the B-spline of degree n with knots $t_i, \cdots, t_{i+n+1}$. For a constant sequence $c = (c_0, \cdots, c_m)$, we let

$$s(x) = \sum_{i=0}^{m} c_i N_{i,n}(x), \quad t_0 < x < t_{m+n+1}. \tag{2.1}$$

In [5], Goodman proved that the bounds on the number of real roots of the spline function

$$z(s) \leq S(c) \tag{2.2}$$

under the following condition

$$Condition(c,t): \ \forall\, x \in (t_0, t_{m+n+1}),\ \exists\, i, \text{ s.t. } t_i < x < t_{i+n+1} \text{ and } c_i \neq 0, \tag{2.3}$$

where $z(s)$ denotes the number of real roots of the spline function $s(x)$, and $S(c)$ denotes the number of sign variations in the sequence c.

Obviously, $Condition(c,t)$ implies that $s(x)$ cannot vanish on any nontrivial interval in (t_0, t_{m+n+1}).

Let us first recall Descartes' rule of signs [7]:

Theorem 2.1. (Descartes' rule of signs) *Let $P(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial in $\mathbb{R}[x]$. If we denote by S(a) the number of sign variations in the sequence $a = (a_0, a_1, \cdots, a_n)$, and $pos(P)$ the number of positive real roots of $P(x)$ counted with multiplicities, then $pos(P) \leq S(a)$, and $pos(P) - S(a)$ is even.*

We remark that Descartes' rule of signs gives the exact number of roots if and only if there is one or no sign variation.

Note that the following direct consequences of sign variations: for any real number sequence $b = (b_i, \cdots, b_j)$, if $b_i b_j > 0$, then b has an even number of sign variations. Moreover, if $b_i b_j < 0$, then b has an odd number of sign variations.

Throughout this paper, we assume $c_j = 0$ when $j < 0$ and $j > m$. For a spline function $s(x)$ defined by (2.1), we have

$$s_i(x) = s(x)\mid_{[t_i, t_{i+1}]} = \sum_{j=i-n}^{i} c_j N_{j,n}(x) \in \mathbf{P}_n,$$

where $\mathbf{P}_n$ denotes the set of all univariate polynomials with real coefficients and degree not exceeding n. Therefore, it can be written in Bézier form:

$$s_i(x) = \sum_{j=0}^{n} b_{i,j} B_{j,n}(t), \quad t \in [0,1] \tag{2.4}$$

under the coordinate transformation

$$t = \frac{x - t_i}{t_{i+1} - t_i}, \quad x \in [t_i, t_{i+1}], \tag{2.5}$$

where $B_{j,n}(t) = C_n^j\, t^j (1-t)^{n-j}$ is the Bernstein polynomial.

Recall that the Bézier curve $s_i(x)$ defined by (2.4) enjoys the variation diminishing property [4]: the curve has no more intersections with any line other than the polygon

$$P_i = \{(\frac{j}{n}, b_{i,j})\}_{j=0}^{n},$$