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CONVERGENCE OF AN IMMERSED INTERFACE UPWIND SCHEME FOR LINEAR ADVECTION EQUATIONS WITH PIECEWISE CONSTANT COEFFICIENTS I: L^1-ERROR ESTIMATES*

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Abstract

We study the L^1-error estimates for the upwind scheme to the linear advection equations with a piecewise constant coefficients modeling linear waves crossing interfaces. Here the interface condition is immersed into the upwind scheme. We prove that, for initial data with a bounded variation, the numerical solution of the immersed interface upwind scheme converges in L^1-norm to the differential equation with the corresponding interface condition. We derive the one-halfth order L^1-error bounds with explicit coefficients following a technique used in [25]. We also use some inequalities on binomial coefficients proved in a consecutive paper [32].

Mathematics subject classification: 65M06, 65M12, 65M25, 35F10.
Key words: Linear advection equations, Immersed interface upwind scheme, Piecewise constant coefficients, Error estimate, Half order error bound.

1. Introduction

In this paper we study the L^1-error estimates for the upwind difference scheme to the linear advection equation

$$\frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = 0, \quad t > 0, \quad x \in \mathbb{R}, \tag{1.1}$$

$$u|_{t=0} = u_0(x), \tag{1.2}$$

with piecewise constant (without loss of generality, a step function in this paper) wave speed

$$c(x) = \begin{cases} c^- & x < 0, \\ c^+ & x > 0. \end{cases} \tag{1.3}$$

Without loss of generality, we assume $c(x) > 0$, which is the local sound speed of the media. At the interface between two different media, c is discontinuous.

Eqs. (1.1)-(1.3) is the simplest case of a hyperbolic equation with singular coefficients. For hyperbolic conservation laws with Lipschitz continuous coefficients, there were numerous works

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on convergence rate estimates for numerical methods. Half-order optimal convergence rates for monotone type or viscosity type methods were established in [2, 24-27, 29]. In contrast, for hyperbolic equations with singular coefficients, or conservation laws with discontinuous flux functions, the convergence rate results for numerical methods are much less studied, although many authors have studied the convergence of the numerical methods. The convergence studies include the convergence of a front tracking method for conservation laws with discontinuous flux functions [4], the convergence of front tracking schemes [3, 5, 18, 19], the Lax-Friedrichs scheme [17] and convergence rate estimates for Godunov's and Glimm's methods [21, 28] for the resonant systems of conservation laws, the convergence of monotone schemes for synthetic aperture radar shape-from-shading equations with discontinuous intensities [23], the convergence of a class of finite difference schemes for the linear conservation equation and the transport equation with discontinuous coefficients [6], the convergence of a difference scheme, based on Godunov or Engquist-Osher flux, for scaler conservation laws with a discontinuous convex flux [30] and the extension to the nonconvex flux [31], the convergence of an upwind difference scheme of Engquist-Osher type for degenerate parabolic convection-diffusion equations with a discontinuous coefficient [16], the convergence of a relaxation scheme for conservation laws with a discontinuous coefficient [15], the convergence of Godunov-type methods for conservation laws with a flux function discontinuous in space [1], the convergence of upwind difference schemes of Godunov and Engquist-Osher type for a scalar conservation law with indefinite discontinuities in the flux function [22]. In the above cases, except for the resonant systems of conservation laws, convergence rates for numerical methods were not studied.

One approach to treat Eqs. (1.1)-(1.3) is to use the equation on domains $x < 0$ and $x > 0$ respectively. Then one needs to provide an interface condition at $x = 0$ to connect the solutions at the two sides of the interface. Once an appropriate interface condition is given, a unique solution of (1.1)-(1.3) can be determined using the method of characteristics. See [12] for the justification of the well-posedness of the Liouville equation with partial transmissions and reflections using this approach in the case of a piecewise constant wave speed with a vertical interface.

The physically relevant interface conditions for (1.1)-(1.3) are not necessarily unique [34]. For example, one can require that u is continuous across the interface,

$$u(0^-, t) = u(0^+, t). \tag{1.4}$$

On the other hand, one can also assume that the flux cu is continuous across the interface,

$$c^- u(0^-, t) = c^+ u(0^+, t). \tag{1.5}$$

Depending on applications, (1.4) and (1.5) are both physically relevant interface conditions being studied. See [34] for more detailed discussions.

A natural and successful approach for computing hyperbolic equations with singular coefficients is to build the interface condition into the numerical scheme. Many efficient numerical methods have been designed using this technique. For example, we mention the immersed interface methods by LeVeque and Li [20, 34].

For Eqs. (1.1)-(1.2) with a general $c(x)$ including indefinite sign changes, the convergence of a class of finite difference schemes to the duality solutions was proved in [6]. For Eqs. (1.1)-(1.3), the duality solution is the one corresponding to the interface condition (1.4). To our knowledge, no error bounds with explicit coefficients have been established for the upwind difference scheme