

ON THE DIVIDED DIFFERENCE FORM OF FAÀ DI BRUNO'S FORMULA II*

Xinghua Wang and Aimin Xu

(Department of Mathematics, Zhejiang University, Hangzhou 310027, China

Email: xwang@zju.edu.cn, xuaimin1009@yahoo.com.cn)

Abstract

In this paper, we consider the higher divided difference of a composite function $f(g(t))$ in which $g(t)$ is an s -dimensional vector. By exploiting some properties from mixed partial divided differences and multivariate Newton interpolation, we generalize the divided difference form of Faà di Bruno's formula with a scalar argument. Moreover, a generalized Faà di Bruno's formula with a vector argument is derived.

Mathematics subject classification: 65D05, 05A10, 41A05.

Key words: Bell polynomial, Faà di Bruno's formula, Mixed partial divided difference, Multivariate Newton interpolation.

1. Introduction

The well-known formula of Faà di Bruno [5] for higher derivative of a composite function plays an important role in combinatorial algebra. The formula together with its representative tools, the Bell polynomials and cycle indicator polynomials of symmetric groups, have wide applications in numerical analysis. Recently, applying Faà di Bruno's formula [10], the coefficients in Lagrangian numerical differentiation formulas and asymptotic expansions of the remainders on local approximation have been derived explicitly. It is proved in [14] that the solutions of a system of equations of algebraic sum of equal powers can be converted to all roots of two univariate algebraic equations whose degree sum equals to the number of unknowns of the underlying system of equations. In [15], we proposed the best quadrature for the function which belongs to any order Sobolev class $KW^r[a, b]$ with the L_∞ norm. In [11], the authors applied the formula again and proved that, for complex polynomials, all extraneous fixed points for any iteration of Halley iterative family (König's algorithms) are repelling. Then six years earlier than [1] they solved the problem left by [9]. Moreover, using the Bell polynomials and cycle indicator polynomials of symmetric groups, a family of parallel and interval iterations for finding all roots of a polynomial simultaneously is established, see, e.g., [16, 17]. These results became the main part of the monograph [8].

Several generalizations of Faà di Bruno's formula for multivariate composite functions have been given in [3, 4, 7]. However, it seems that the details of the proofs and expressions are so cumbersome that they are difficult to be used for practical computations. In a recent paper [13], we established the divided difference form of Faà di Bruno's formula which is of simple form. It happened that Floater [6] also derived a similar result independently. It is the purpose of this paper to continue the above work and to give the high order divided difference of h which is represented by the divided differences of f and g . The functions f and g satisfy $f : V \rightarrow \mathbb{F}$, $g : U \rightarrow V$, and the function $h : U \rightarrow \mathbb{F}$ is the composite function $f \circ g$, which is denoted

* Received April 26, 2006; final revised August 29, 2006; accepted June 29, 2006.

by $t \mapsto f(g(t))$. Here \mathbb{F} denotes the set of real numbers \mathbb{R} or the set of complex numbers \mathbb{C} . Moreover, $U \subset \mathbb{F}$, $V \subset \mathbb{F}^s$ and s is a positive integer. We follow the definition of the divided difference of g from [2]. So we can obtain the following generalized Faà di Bruno’s formula,

$$\frac{1}{n!}h^{(n)}(t) = \sum_{1 \leq |\alpha| \leq n} \frac{1}{|\alpha|!} f^{(\alpha)}(g(t)) \hat{B}_{n,\alpha} \left(\frac{g'(t)}{1!}, \frac{g''(t)}{2!}, \dots, \frac{g^{(n)}(t)}{n!} \right), \tag{1.1}$$

where

$$\hat{B}_{n,\alpha}(x_1, x_2, \dots, x_n) := \sum_{e_{i_1} + \dots + e_{i_{|\alpha|}} = \alpha} \sum_{\substack{k_1 + \dots + k_{|\alpha|} = n \\ k_1, \dots, k_{|\alpha|} \geq 1}} \prod_{j=1}^{|\alpha|} x_{k_j}^{e_{i_j}} \tag{1.2}$$

is the ordinary partial Bell polynomial with several vector variables x_1, x_2, \dots, x_n . We will introduce all notations in the next section. However, in order to understand formula (1.1) well, here we present an example for the generalized Faà di Bruno’s formula. Let $h(t) = f(x(t), y(t))$. Then we have

$$\begin{aligned} \frac{h^{(4)}(t)}{4!} &= \frac{\partial f}{\partial x} \frac{x^{(4)}(t)}{4!} + \frac{\partial f}{\partial y} \frac{y^{(4)}(t)}{4!} \\ &+ \frac{1}{2!} \frac{\partial^2 f}{\partial x^2} \left(\frac{1}{3} x'(t)x'''(t) + \frac{1}{3} x''(t)^2 \right) + \frac{1}{2!} \frac{\partial^2 f}{\partial y^2} \left(\frac{1}{3} y'(t)y'''(t) + \frac{1}{4} y''(t)^2 \right) \\ &+ \frac{1}{2!} \frac{\partial^2 f}{\partial x \partial y} \left(\frac{1}{3} x'(t)y'''(t) + \frac{1}{2} x''(t)y''(t) + \frac{1}{3} x'''(t)y'(t) \right) \\ &+ \frac{1}{3!} \frac{\partial^3 f}{\partial x^3} \left(\frac{3}{2} x'(t)^2 x''(t) \right) + \frac{1}{3!} \frac{\partial^3 f}{\partial x^2 \partial y} \left(\frac{3}{2} x'(t)^2 y''(t) + 3x'(t)x''(t)y'(t) \right) \\ &+ \frac{1}{3!} \frac{\partial^3 f}{\partial x \partial y^2} \left(3x'(t)y'(t)y''(t) + \frac{3}{2} x''(t)y'(t)^2 \right) + \frac{1}{3!} \frac{\partial^3 f}{\partial y^3} \left(\frac{3}{2} y'(t)^2 y''(t) \right) \\ &+ \frac{1}{4!} \frac{\partial^4 f}{\partial x^4} x'(t)^4 + \frac{1}{4!} \frac{\partial^4 f}{\partial x^3 \partial y} (4x'(t)^3 y'(t)) + \frac{1}{4!} \frac{\partial^4 f}{\partial x^2 \partial y^2} (6x'(t)^2 y'(t)^2) \\ &+ \frac{1}{4!} \frac{\partial^4 f}{\partial x \partial y^3} (4x'(t)y'(t)^3) + \frac{1}{4!} \frac{\partial^4 f}{\partial y^4} y'(t)^4. \end{aligned}$$

2. Preliminaries

In order to simplify expressions, it is convenient to recall some multivariate notations. We denote by \mathbb{Z}_+ the set of nonnegative integers and by \mathbb{Z}_+^s the set of multi-integers. $e_i \in \mathbb{Z}_+^s$ is a unit vector whose j th component is δ_{ij} where

$$\delta_{ij} := \begin{cases} 0, & j \neq i; \\ 1, & j = i. \end{cases}$$

Let \mathbb{F}^s be the s -dimensional Euclidean space and $x \in \mathbb{F}^s$ be an s -dimensional vector whose i th component is denoted by x^{e_i} . $\alpha \in \mathbb{Z}_+^s$ denotes a multi-index. $|\alpha| := \sum_{i=1}^s \alpha^{e_i}$ is its length. More generally,

$$(\cdot)^\alpha : \mathbb{F}^s \rightarrow \mathbb{F} : x \mapsto x^\alpha := \prod_{i=1}^s (x^{e_i})^{\alpha^{e_i}},$$