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A ROBUST FINITE ELEMENT METHOD FOR A 3-D ELLIPTIC SINGULAR PERTURBATION PROBLEM*

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Abstract

This paper proposes a robust finite element method for a three-dimensional fourth-order elliptic singular perturbation problem. The method uses the three-dimensional Morley element and replaces the finite element functions in the part of bilinear form corresponding to the second-order differential operator by a suitable approximation. To give such an approximation, a convergent nonconforming element for the second-order problem is constructed. It is shown that the method converges uniformly in the perturbation parameter.

Mathematics subject classification: 65N30.
Key words: Finite element, Singular perturbation problem.

1. Introduction

Let Ω be a bounded polyhedral domain of R^n with $1 \leq n \leq 3$. Denote the boundary of Ω by $\partial\Omega$. For $f \in L^2(\Omega)$, we consider the following boundary value problem of the fourth-order elliptic singular perturbation equation:

$$\begin{cases} \varepsilon^2 \Delta^2 u - \Delta u = f, & \text{in } \Omega, \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $\nu = (\nu_1, \cdots, \nu_n)^\top$ is the unit outer normal of $\partial\Omega$, Δ is the standard Laplacian operator and ε is a small parameter satisfying $0 < \varepsilon \leq 1$. When $\varepsilon \to 0$ the differential equation formally degenerates to the Poisson equation.

In the two-dimensional case, the Morley element was proposed in [9] for the plate bending problem. The Morley element is convergent for fourth-order elliptic problems, but is divergent for second-order problems (see, e.g., [5, 8, 13]). The Morley element and an C^0 modified Morley element for problem (1.1) were discussed in [10]. It was shown that the modified Morley element is uniformly convergent with respect to ε while the Morley element does not converge when $\varepsilon \to 0$. Two non-C^0 nonconforming elements were proposed in [4] by the double set parameter technique. These two elements were also proved to be uniformly convergent. A modified Morley element method for problem (1.1) was proposed in [15]; it is convergent uniformly with respect to ε. This method also uses the Morley element (or the rectangle Morley element), but the linear approximation (or the bilinear approximation) of finite element functions is used in the part of the bilinear form corresponding to the second-order differential term.

In this paper, we consider the three-dimensional case. The three-dimensional Morley element can be found in [11] or in [14]. We will take a similar way used in [15] and propose a modified

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Morley element method for problem (1.1). We will use certain approximation of finite element functions in the part of the bilinear form corresponding to the second-order differential term. It will be shown that the modified method converges uniformly in the perturbation parameter ε. The three-dimensional Morley element uses the integral averages of the function over all edges as degrees of freedom instead of the function values at vertices. To given suitable approximation of the finite element function, we need to construct a convergent nonconforming finite element for the Poisson equation with the integral averages of the function over all edges as degrees of freedom.

Problem (1.1) is a boundary value problem of a stationary linearizing form of the Cahn-Hilliard equation. The modelling in material science makes use of the Cahn-Hilliard equations in three dimensions (see, e.g., [2, 3, 6]). Besides the theoretical interest, our new finite element method is expected to be useful in the computation of the Cahn-Hilliard equation.

The paper is organized as follows. The rest of this section lists some preliminaries. Section 2 describes a nonconforming finite element for the Poisson equation. Section 3 gives the detailed descriptions of the modified Morley element method. Section 4 shows the uniform convergence of the method.

Throughout this paper, we assume $n = 3$. For a nonnegative integer s, let $H^s(\Omega)$, $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Omega}$ denote the usual Sobolev space, norm and semi-norm, respectively. Let $H_0^s(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$ with respect to the norm $\|\cdot\|_{s,\Omega}$ and $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)$. Define

$$a(v,w) = \int_\Omega \sum_{i,j=1}^{3} \frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial^2 w}{\partial x_i \partial x_j}, \quad \forall v, w \in H^2(\Omega), \tag{1.2}$$

$$b(v,w) = \int_\Omega \sum_{i=1}^{3} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i}, \quad \forall v, w \in H^1(\Omega). \tag{1.3}$$

The weak form of problem (1.1) is: find $u \in H_0^2(\Omega)$ such that

$$\varepsilon^2 a(u,v) + b(u,v) = (f,v), \quad \forall v \in H_0^2(\Omega). \tag{1.4}$$

Let u^0 be the solution of following boundary value problem:

$$\begin{cases} -\Delta u^0 = f, & \text{in } \Omega, \\ u^0|_{\partial\Omega} = 0. \end{cases} \tag{1.5}$$

For a mesh size h, let $\mathcal{T}_h$ be a triangulation of Ω consisting of tetrahedra. For each $T \in \mathcal{T}_h$, let h_T be the diameter of the smallest ball containing T and ρ_T be the diameter of the largest ball contained in T. Let $\{\mathcal{T}_h\}$ be a family of triangulations with $h \to 0$. Throughout the paper, we assume that $h_T \le h \le \eta\rho_T$, $\forall T \in \mathcal{T}_h$, with η a positive constant independent of h.

2. A Nonconforming Element for the Poisson Equation

For a subset $B \subset R^3$ and a nonnegative integer r, let $P_r(B)$ be the space of all polynomials with degree not greater than r.

Given a tetrahedron T, its four vertices are denoted by a_j, $1 \le j \le 4$. The face of T opposite a_j is denoted by F_j, $1 \le j \le 4$. The edge with a_i and a_j as its vertices, is denoted by S_{ij},