Journal of Computational Mathematics, Vol.25, No.5, 2007, 620–630.

# AN INVERSE EIGENVALUE PROBLEM FOR JACOBI MATRICES *

Haixia Liang and Erxiong Jiang
(*Department of Mathematics, Shanghai University, Shanghai 200444, China*
*Email: lianghaixia1980@163.com, ejiang@fudan.edu.cn*)

**Abstract**

In this paper, we discuss an inverse eigenvalue problem for constructing a $2n \times 2n$ Jacobi matrix $T$ such that its $2n$ eigenvalues are given distinct real values and its leading principal submatrix of order $n$ is a given Jacobi matrix. A new sufficient and necessary condition for the solvability of the above problem is given in this paper. Furthermore, we present a new algorithm and give some numerical results.

*Mathematics subject classification:* 65L09.
*Key words:* Symmetric tridiagonal matrix, Jacobi matrix, Eigenvalue problem, Inverse eigenvalue problem.

## 1. Introduction

A real symmetric tridiagonal matrix $T_{1,n}$ of the form

$$T_{1,n} = \begin{pmatrix} \alpha_1 & \beta_1 & & & 0 \\ \beta_1 & \ddots & & & \\ & \ddots & & & \\ 0 & & & \beta_{n-1} & \alpha_n \end{pmatrix}$$

with $\beta_i > 0$ is called a Jacobi matrix.

In 1979, Hochstand [1] put forward the inverse eigenvalue problem (I): Given a Jacobi matrix $T_n$ and real values: $\lambda_1 < \lambda_2 < \cdots < \lambda_{2n}$, construct an irreducible symmetric tridiagonal matrix $T_{1,2n}$ whose eigenvalues are $\lambda_1, \lambda_2, \cdots, \lambda_{2n}$ and the leading principal submatrix $T_{1,n}$ is the given $T_n$.

Hochstand also proved that the solution is unique if it exists. In 1987, Boley and Golub [2] proposed a numerical method for solving Problem (I), but this method needs to compute all the eigenvalues and eigenvectors of $T_{1,n}$, which seems expensive in computational time. Dai [3] gave a sufficient and necessary condition for solving this problem, which was further improved by Xu [4]. But both algorithms need to compute $2n+1$ determinants of matrices of order $2n$. Furthermore, in the process of constructing $T_{1,2n}$, we find that $T_{1,n}$ is reconstructed, which may make $T_{1,n}$ different from the given one due to the computing error. In this paper, the inverse problem is solved by an idea completely different from the previous ones. In fact, since $T_{1,n}$ is given , we may only take measures to obtain $T_{n+1,2n}$ and $\beta_n$.

In this paper, we present a new algorithm based on the following $(k)$ Jacobi inverse eigenvalue problem [5]: Given real number sets $S_1 = \{\mu_1, \cdots, \mu_{k-1}\}$, $S_2 = \{\mu_{k+1}, \cdots, \mu_n\}$ *and* $S_3 =$

* Received September 26, 2005; final revised February 16, 2006; accepted June 29, 2006.

$\{\lambda_1, \cdots, \lambda_n\}$, we construct $T_{1,n}$ of the form

$$T_{1,n} = \begin{pmatrix} T_{1,k-1} & \beta_{k-1} & \\ \beta_{k-1} & \alpha_k & \beta_k \\ & \beta_k & T_{k+1,n} \end{pmatrix},$$

where the eigenvalue sets of $T_{1,k-1}, T_{k+1,n}$ and $T_{1,n}$ are $S_1, S_2$ and $S_3$, respectively.

Let

$$T_{1,2n} = \begin{pmatrix} \alpha_1 & \beta_1 & & & & & 0 \\ \beta_1 & \ddots & \ddots & & & & \\ & \ddots & & \beta_{n-1} & & & \\ & & \beta_{n-1} & \alpha_n & \beta_n & & \\ & & & \beta_n & \ddots & \ddots & \\ & & & & \ddots & \ddots & \beta_{2n-1} \\ 0 & & & & & \beta_{2n-1} & \alpha_{2n} \end{pmatrix}$$

be a $2n \times 2n$ irreducible symmetric tridiagonal matrix, and denote its submatrix $T_{p,q}$ $(p < q)$ by

$$T_{p,q} = \begin{pmatrix} \alpha_p & \beta_p & & & 0 \\ \beta_p & \alpha_{p+1} & \beta_{p+1} & & \\ & \beta_{p+1} & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_{q-1} \\ 0 & & & \beta_{q-1} & \alpha_q \end{pmatrix}.$$

Here, we assume that $T_{1,2n}$ and $T_{p,q}$ are Jacobi matrices. Rewrite

$$T_{1,2n} = \begin{pmatrix} T_{1,n-1} & \beta_{n-1} & \\ \beta_{n-1} & \alpha_n & \beta_n \\ & \beta_n & T_{n+1,2n} \end{pmatrix},$$

where $\beta_n$ and $T_{n+1,2n}$ in problem (I) need to be obtained from the given values $\{\lambda_i\}_{i=1}^{2n}$ and the matrix $T_{1,n}$.

The paper is organized as follows. In Section 2, a sufficient and necessary condition for solving Problem (I) is given in two cases: $T_{1,n-1}$ and $T_{1,2n}$ have or do not have common eigenvalues. We also prove that if the solution exists, then it is unique. In Section 3, we present the corresponding algorithm and give two numerical examples.

## 2. The Basic Theorems

### 2.1. Some basic lemmas

In this section, we first give some preliminary results which play a fundamental role in this paper. The proofs are omitted here; they can be found in the corresponding references.

**Lemma 2.1.** [6] *Let the eigenvalues of $T_n$ be $\theta_1 < \theta_2 < \cdots < \theta_n$, with corresponding unit eigenvectors $S_1$, $S_2, \cdots, S_n$. Denote the i-th component of $S_j$ $(j = 1, 2, \cdots, n)$ by $S_{ij}$. Then, for $\mu \le \nu$,*

$$\chi'_{1,n}(\theta_j) S_{\mu j} S_{\nu j} = \chi_{1,\mu-1}(\theta_j) \beta_\mu \cdots \beta_{\nu-1} \chi_{\nu+1,n}(\theta_j),$$