Journal of Computational Mathematics, Vol.25, No.5, 2007, 543–552.

OPTIMAL APPROXIMATE SOLUTION OF THE MATRIX EQUATION $AXB = C$ OVER SYMMETRIC MATRICES *

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Abstract

Let S_E denote the least-squares symmetric solution set of the matrix equation $AXB = C$, where A, B and C are given matrices of suitable size. To find the optimal approximate solution in the set S_E to a given matrix, we give a new feasible method based on the projection theorem, the generalized SVD and the canonical correction decomposition.

Mathematics subject classification: 15A24, 65F20, 65F22, 65K10.
Key words: Least-squares solution, Optimal approximate solution, Generalized singular value decomposition, Canonical correlation decomposition.

1. Introduction

Denote by $R^{m\times n}$ the set of real $m \times n$ matrices, and $SR^{n\times n}$ the set of symmetric matrices in $R^{n\times n}$. In this paper, we consider the following problem:

Problem 1.1. *Given $A \in R^{m\times n}$, $B \in R^{n\times p}$, $C \in R^{m\times p}$ and $X^* \in SR^{n\times n}$. Let*

$$S_E = \{X|X \in SR^{n\times n},\ \|AXB - C\| = \min_{Y\in SR^{n\times n}} \|AYB - C\|\}.$$

Find $\widehat{X} \in S_E$ such that

$$\|\widehat{X} - X^*\| = \min_{X\in S_E} \|X - X^*\|,$$

where $\|\cdot\|$ denotes the Frobenius norm.

In other word, S_E is the least-squares symmetric solution set of the matrix equation

$$AXB = C, \tag{1.1}$$

and $\widehat{X}$ is the optimal approximate least-squares symmetric solution of the matrix equation (1.1) to the given matrix X^*.

The consistency conditions of the matrix equation (1.1) with the symmetric solution were given by Chu [1] (see also Dai [3]), and the symmetric solutions can also be obtained by using the generalized singular value decomposition (GSVD) when the matrix equation is consistent. For the matrix equation (1.1), Wang and Chang [17] gave the least-squares symmetric solution by using GSVD; Liao and Bai [12] and Deng [5] considered the least-squares solution over the symmetric positive semi-definite matrices and positive semi-definite matrices, respectively; and Yuan [19] also gave the minimum-norm least-squares symmetric solution for the consistent matrix equation (1.1) by using the canonical correlation decomposition (CCD).

* Received December 15, 2005; final revised September 1, 2006; accepted October 1, 2006.

The problem of finding a nearest matrix in the least-squares symmetric solution set of a matrix equation to a given matrix in the sense of the Frobenius norm, that is, Problem 1.1 in this paper, is called the matrix nearness problem. The matrix nearness problem is initially proposed in the processes of test or recovery of linear systems due to incomplete dates or revising dates. A preliminary estimate X^* of the unknown matrix X can be obtained by the experimental observation values and the information of statical distribution. There are many important results on the discussions of the matrix nearness problem associated with other matrix equations, we refer the reader to [2, 4, 8, 9, 10, 15] and references therein.

In this paper, we develop an efficient method to solve Problem 1.1. Our approach is based on the projection theorem in Hilbert space, GSVD and CCD of matrix pairs. It can be essentially divided into three parts: First, we find a least-squares solution X_0 of the matrix equation (1.1) by using GSVD; then utilizing the solution X_0 and the projection theorem, we transfer Problem 1.1 to a problem of finding the optimal approximate symmetric solution of a consistent matrix equation; finally, we find the optimal approximate symmetric solution of the consistent matrix equation by using CCD.

The paper is organized as follows. After introducing some necessary notations and several useful lemmas in Section 2, we will discuss Problem 1.1 in Section 3, and give the expression of its solution. Then, in Section 4, we give the numerical algorithm to compute the solution of Problem 1.1. Numerical experiments will be carried out in Section 4.

2. Notations and Lemmas

The notation used in this paper can be summarized as follows: the set of all $n\times n$ orthogonal matrices in $R^{n\times n}$ is denoted by $OR^{n\times n}$. Denote by I the unit matrix. A^T, $tr(A)$ and $rank(A)$ respectively denote the transpose, the trace and the rank of the matrix A. For $A=(a_{ij})\in R^{m\times n}$, $B=(b_{ij})\in R^{m\times n}$, $A*B$ represents the Hadamard product of the matrices A and B, that is, $A*B=(a_{ij}b_{ij})_{m\times n}$. Let $\langle A,B\rangle$ represent the inner product of the matrices A and B, that is, $\langle A,B\rangle = tr(B^TA)$. Then $R^{m\times n}$ is a Hilbert inner product space, and the norm of a matrix produced by the inner product is the Frobenius norm.

We first state the concepts of the GSVD and CCD, which are essential tools for deriving the solution of Problem 1.1. See [6, 7, 11, 13, **?**, 16] for details.

Let $A\in R^{m\times n}$ and $B\in R^{n\times p}$. Then the GSVD of the matrix pair (A,B^T) is given by

$$A=U\Sigma_A M \quad \text{and} \quad B^T=V\Sigma_B M, \tag{2.1}$$

where $U\in OR^{m\times m}$ and $V\in OR^{p\times p}$; $M\in R^{n\times n}$ is a nonsingular matrix; and

$$\Sigma_A=\begin{pmatrix} I_r & 0 & 0 & 0\\ 0 & S_A & 0 & 0\\ 0 & 0 & 0_{(m-r-s)\times(k-r-s)} & 0\end{pmatrix} \quad \text{and} \quad \Sigma_B=\begin{pmatrix} 0_{(p+r-k)\times r} & 0 & 0 & 0\\ 0 & S_B & 0 & 0\\ 0 & 0 & I_{(k-r-s)} & 0\end{pmatrix}$$

are block matrices, with the diagonal matrices S_A and S_B being given by

$$S_A=diag(\alpha_1,\alpha_2,\cdots,\alpha_s)>0 \quad \text{and} \quad S_B=diag(\beta_1,\beta_2,\cdots,\beta_s)>0.$$

Here

$$k=rank(A^T,B),\quad r=k-rank(B),\quad s=rank(A)+rank(B)-k.$$