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# EDGE-ORIENTED HEXAGONAL ELEMENTS *1)

Chao Yang and Jiachang Sun
(*Parallel Computing Lab., Institute of Software, Chinese Academy of Sciences, Beijing 100080, China*
*Email: yc@mail.rdcps.ac.cn, sun@mail.rdcps.ac.cn*)

**Abstract**

In this paper, two new nonconforming hexagonal elements are presented, which are based on the trilinear function space $Q_1^{(3)}$ and are edge-oriented, analogical to the case of the rotated $Q_1$ quadrilateral element. A priori error estimates are given to show that the new elements achieve first-order accuracy in the energy norm and second-order accuracy in the $L^2$ norm. This theoretical result is confirmed by the numerical tests.

*Mathematics subject classification:* 65N15, 65N30.
*Key words:* Nonconforming finite element method, Hexagonal element, $Q_1$ element.

## 1. Introduction

The finite element method (FEM) is a powerful tool, which can be easily applied to a large variety of engineering applications. In two dimensions, classical FEMs often treat meshes consisting of triangles, quadrilaterals, etc. While as is well-known, hexagons also extensively exist in the nature as well as in some special application fields, such as in material sciences and nuclear engineering [3, 12, 13]. Moreover, besides triangles and quadrilaterals, only hexagons can form a regular tessellation of the plane [4], which inspires us to consider hexagonal elements.

Noticing that a bivariate quadratic polynomial has six degree of freedoms, one may ask whether the six vertices of a hexagon exactly determine a bivariate quadratic polynomial. Unfortunately, the resulting equation is not unisolvable in general, since the six vertices of the regular hexagon belong to a same quadratic curve, a circle. To construct conforming hexagonal elements avoiding polynomial spaces, some works based on rational function spaces have been carried out in [10, 12, 13, 17]. Moreover, while the nonconforming triangular and quadrilateral elements are well studied, see, e.g., [7, 11, 14, 15, 16], their hexagonal counterparts are less complete. This motivates us to study nonconforming hexagonal elements.

The main goal of this paper is to generalize the quadrilateral rotated $Q_1$ element [14] to the hexagonal case. We use the so-called three-directional coordinates [18] to explore the symmetry of a hexagon. Two new elements are constructed, both of which are based on trilinear function space $Q_1^{(3)}$ and are edge-oriented. The modified version has an extra degree of freedom on the element face, which is similar to the five-node element proposed by Han in [11]. Optimal order error estimates are given with respect to the energy norm and the $L^2$ norm. Numerical experiments are presented to demonstrate the accuracy of the proposed method.

Before the end of this section, we recall some notations (or refer to [1, 2]). Let $(\cdot,\cdot)$ denote the $L^2$ inner product and $||\cdot||_{H^p(\Omega)}$ (resp. $|\cdot|_{H^p(\Omega)}$) be the norm (resp. semi-norm) for the Sobolev space $H^p(\Omega)$.

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## 2. Nonconforming Hexagonal Element

To begin, we introduce the three-directional coordinates with which the symmetries of a regular hexagon $\widehat{H}$ could be well embodied. As is well-known, under Cartesian coordinates, a plane can be viewed as $\{(t_1, t_2, t_3) \mid t_3 = 0\}$ in the space. While under the three-directional coordinates, the plane $S = \{(t_1, t_2, t_3) \mid t_1 + t_2 + t_3 = 0\}$ are studied. For more details, we refer to [18]. Thus any point in the plane $S$ can be represented by a coordinates triple $(t_1, t_2, t_3)$ with $t_1 + t_2 + t_3 = 0$. A natural coordinates transform between Cartesian coordinates and three-directional coordinates can be

$$\begin{cases} \xi = \dfrac{1}{2}(t_3 - t_2), \\ \\ \eta = \dfrac{\sqrt{3}}{2} t_1, \end{cases} \qquad \text{and} \qquad \begin{cases} t_1 = \dfrac{2}{\sqrt{3}}\eta, \\ t_2 = -\xi - \dfrac{1}{\sqrt{3}}\eta, \\ t_3 = \xi - \dfrac{1}{\sqrt{3}}\eta. \end{cases}$$

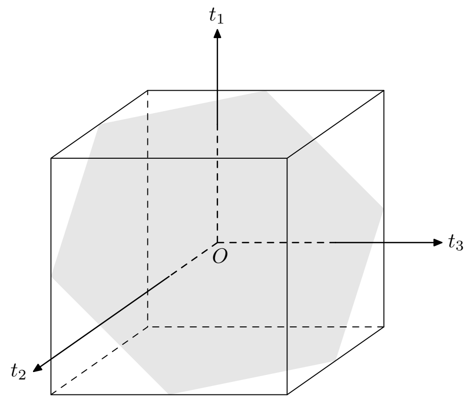


Fig. 2.1. Getting a regular hexagon from a unit-cube.

We let $B = \{(t_1, t_2, t_3) \mid -1 < t_1, t_2, t_3 < 1\}$ be a box domain in the space. Then as illustrated in Fig. 2.1, the regular hexagon $\widehat{H}$ can be easily obtained by letting $\widehat{H} = B \cap S$. Denote the trilinear space over $\widehat{H}$ as

$$Q_1^{(3)}(\widehat{H}) = \text{span}\{1,\ t_1,\ t_2,\ t_3,\ t_2t_3,\ t_3t_1,\ t_1t_2,\ t_1t_2t_3\};$$

obviously we have $\dim(Q_1^{(3)}(\widehat{H})) = 2^3 - 1 = 7$.

We refer symmetric parallel hexagons as an affine-equivalence class of the regular hexagon. For a symmetric parallel hexagon, any two opposite sides are parallel and the three main diagonals meet at one symmetric point, see Fig. 2.2.

For simplicity, assume that $\Omega$ is a polygon domain and $\mathcal{T}_h$ be a decomposition of $\Omega$ consisted by symmetric parallel hexagons and triangles, where $h = \max_{K \in \mathcal{T}_h} \text{diam} K$. By $\partial\mathcal{T}_h$ we denote the set of all edges $F$ of the element $K \in \mathcal{T}_h$. Assume $\mathcal{T}_h$ satisfies the usual "quasi-uniform" condition [1, 2]. Accordingly, the generic constant $C$ used below is always independent of $h$. We take the unit regular hexagon $\widehat{H}$ and the unit equilateral triangle $\widehat{T}$ as the reference element. For any $K \in \mathcal{T}_h$, there exists a unique and invertible affine map $F_K : \widehat{K} \to K$, $F_K = B_K\widehat{x} + b_K := x$, where $\widehat{K}$ could be $\widehat{H}$ or $\widehat{T}$.