Journal of Computational Mathematics, Vol. 25, No. 3, 2007, 368–373.

INITIAL VALUE TECHNIQUES FOR THE HELMHOLTZ AND MAXWELL EQUATIONS*

Frank Natterer and Olga Klyubina
(*Department of Mathematics, University of Münster, Einsteinstrasse 62, 48149 Münster, Germany*
Email: nattere@math.uni-muenster.de, klyubina@math.uni-muenster.de)

Abstract

We study the initial value problem of the Helmholtz equation with spatially variable wave number. We show that it can be stabilized by suppressing the evanescent waves. The stabilized Helmholtz equation can be solved numerically by a marching scheme combined with FFT. The resulting algorithm has complexity $n^2 \log n$ on a $n \times n$ grid. We demonstrate the efficacy of the method by numerical examples with caustics. For the Maxwell equation the same treatment is possible after reducing it to a second order system. We show how the method can be used for inverse problems arising in acoustic tomography and microwave imaging.

Mathematics subject classification: 35J05, 65N21.
Key words: Stability of elliptic initial value problems, Parabolic wave equation, Inverse problems in acoustics and electromagnetics.

1. Introduction

The initial value problem for elliptic equations, such as the Helmholtz and the Maxwell equations, are notoriously unstable. There exists a huge literature on stabilizing these initial value problems. Common features of these works are the use of a-priori information about the exact solution and conditional stability estimates; see [1]. For a recent paper that provides an overview and the spirit of these works see [2].

In this paper we follow a completely different route. We consider a differential equation of the form

$$\Delta u + k^2(1 + f(x))u = 0. \tag{1.1}$$

For a large parameter k we show that the Cauchy initial value problem for this equation is perfectly stable, provided we restrict ourselves to low frequencies, *i.e.*, the part of the solution u that is obtained by low-pass filtering u with a cut-off frequency near k. In other words, the instability is a pure high frequency phenomenon and disappears as soon as the high frequencies are removed. We do not need a-priori assumptions, and our estimates are linear. Physically our stabilization means the removal of the evanescent waves.

Estimates of this type were derived in [7] for the Helmholtz equation and in [6] for the Maxwell equations by energy estimates. These estimates contain powers of order 2 and even 4 of k which make the application to high frequency imaging questionable. In Section 2 we derive new estimates with a much better behavior in terms of k. In fact they have negative powers of k. These new estimates are based on the thesis [10]. They can be viewed as the analogue of the

* Received October 31, 2006; Final revised November 21, 2006.

famous $1/k$ estimates for the inverse Helmholtz operator of [4] and [3]; they are also reminiscent of the recent work [9]. In Section 3 we give numerical examples for initial value problems with a focal point. In Section 4 we demonstrate the usefulness of the initial value approach to inverse problems.

2. Stability Estimates

For simplicity we restrict ourselves to the Helmholtz case with zero initial values, *i.e.*, we consider the initial value problem

$$\Delta u + k^2(1+f)u = r, \quad x_2 > 0, \quad u(x_1,0) = 0, \quad \frac{\partial u}{\partial x_2}(x_1,0) = 0. \tag{2.1}$$

Theorem 2.1. *Let $f \in C^1(\mathbb{R}^2)$ be real valued and supported in $[-\rho,\rho]\times[0,\infty]$, and let m be a constant such that $-1 < m \le f$. Then, for $\kappa = \theta k\sqrt{1+m}, 0<\theta<1$, there exists a constant c such that*

$$||u_{k\vartheta}(\cdot,x_2)||_{L_2(-\rho,\rho)} \le \frac{\sqrt{\rho}e^{\rho c}}{k\vartheta}||r||_{L_2(-\rho,\rho)\times(0,\rho)}, \tag{2.2}$$

where

$$\vartheta = \sqrt{1+m}\sqrt{1-\theta^2}. \tag{2.3}$$

Proof. In a first step we assume f to be piecewise constant as a function of x_2, *i.e.*,

$$f(x_1,x_2) = f_i(x_1), \quad ih \le x_2 \le (i+1)h$$

with some $h>0$. Fourier transforming (2.1) with respect to x_1 yields for $ih \le x_2 \le (i+1)h$

$$\frac{d^2}{dx_2^2}\hat{u}(\cdot,x_2) + A_i\hat{u}(\cdot,x_2) = \hat{r}(\cdot,x_2), \tag{2.4}$$

the operator A_i in $L_2(\mathbb{R}^1)$ being defined by

$$(A_iv)(\xi_1) = (k^2-\xi_1^2)v(\xi_1) + (2\pi)^{-1/2}k^2(\hat{f}_i * v)(\xi_1)$$

with * the convolution in $\mathbb{R}^1$. Since f is real, A_i is selfadjoint . We have by Parseval's relation

$$\begin{aligned}(\hat{f}_i * v, v)_{L_2(\mathbb{R}^1)} &= \int_{-\infty}^{+\infty}(\hat{f}_i * v)\bar{v}d\xi_1 = \int_{-\infty}^{+\infty}\widetilde{(\hat{f}_i * v)}\;\bar{\tilde{v}}dx_1 \\ &= (2\pi)^{1/2}\int_{-\infty}^{+\infty} f_i|\tilde{v}|^2dx_1 \ge (2\pi)^{1/2}m(v,v)_{L^2(\mathbb{R}^1)}.\end{aligned}$$

Applying this to functions v supported in $[-\kappa,\kappa]$ we obtain for the restriction of A_i to $L_2(-\kappa,\kappa)$ (again denoted by A_i)

$$(A_iv,v)_{L_2(-\kappa,\kappa)} \ge (k^2-\kappa^2+k^2m)(v,v)_{L_2(-\kappa,\kappa)}.$$

Integrating (2.4) over $[ih,x_2]$ we obtain

$$\begin{aligned}\hat{u}(\cdot,x_2) = &\cos(K_i(x_2-ih))\,\hat{u}(\cdot,x_2) + K_i^{-1}\sin K_i(x_2-ih)\frac{\partial\hat{u}}{\partial\xi_1}(\cdot,x_2) \\ &+\int_{ih}^{x_2} K_i^{-1}\sin(K_i(x_2-x_2'))\hat{r}(\cdot,x_2')dx_2',\end{aligned} \tag{2.5}$$