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MINIMIZATION PROBLEM FOR SYMMETRIC ORTHOGONAL ANTI-SYMMETRIC MATRICES [*1)]

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Abstract

By applying the generalized singular value decomposition and the canonical correlation decomposition simultaneously, we derive an analytical expression of the optimal approximate solution $\widehat{X}$, which is both a least-squares symmetric orthogonal anti-symmetric solution of the matrix equation $A^TXA = B$ and a best approximation to a given matrix X^*. Moreover, a numerical algorithm for finding this optimal approximate solution is described in detail, and a numerical example is presented to show the validity of our algorithm.

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Key words: Symmetric orthogonal anti-symmetric matrix, Generalized singular value decomposition, Canonical correlation decomposition.

1. Introduction

Denote the set of all symmetric (anti-symmetric) matrices in $\mathcal{R}^{n\times n}$ by $\mathcal{S}^{n\times n}(\mathcal{A}^{n\times n})$, the set of all orthogonal matrices in $\mathcal{R}^{n\times n}$ by $\mathcal{O}^{n\times n}$, the $n\times n$ identity matrix by I_n, the transpose and the Frobenius norm of a real matrix A by A^T and $\|A\|$, respectively. For $A = (a_{ij}) \in \mathcal{R}^{n\times m}$, $B = (b_{ij}) \in \mathcal{R}^{n\times m}$, $A * B$ represents the Hadamard product of the matrices A and B, that is, $A * B = (a_{ij}b_{ij}) \in \mathcal{R}^{n\times m}$. Let $\mathcal{SOR}^{n\times n}$ be the set of all real $n \times n$ symmetric orthogonal matrices, i.e., $\mathcal{SOR}^{n\times n} = \{P|P = P^T, P^2 = I_n, P \in \mathcal{R}^{n\times n}\}$.

Definition 1.1. *Given $P \in \mathcal{SOR}^{n\times n}$ and let $X \in \mathcal{S}^{n\times n}$.*

(1) The matrix X is called symmetric orthogonal symmetric with respect to P if $(PX)^T = PX$. The set of all $n \times n$ symmetric orthogonal symmetric matrices is denoted by $\mathcal{SS}_P^{n\times n}$;

(2) The matrix X is called symmetric orthogonal anti-symmetric matrix with respect to P if $(PX)^T = -PX$. The set of all symmetric orthogonal anti-symmetric matrices is denoted by $\mathcal{SA}_P^{n\times n}$.

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The symmetric orthogonal (anti-)symmetric matrices play an important role in numerical analysis and matrix theory. For example, the matrix $X \in \mathcal{SS}_P^{n\times n}(\mathcal{SA}_P^{n\times n})$ can preserve the symmetric (anti-symmetric) structure after applying a Householder transformation, because the Householder matrix is symmetric and orthogonal. Let $J_n = [e_n, e_{n-1}, \ldots, e_1]$, where e_i denote the ith column of I_n. It is easy to verify that $J_n \in \mathcal{SOR}^{n\times n}$. If $P = J_n$, then $\mathcal{SS}_P^{n\times n}$ and $\mathcal{SA}_P^{n\times n}$ are a bi-symmetric matrix set [5, 12, 19] and a symmetric and skew anti-symmetric matrix set [18, 22], respectively, which have been applied in various areas [18, 24], such as information theory, linear system theory and numerical analysis. If $P = I_n$, then $\mathcal{SS}_P^{n\times n}$ is a symmetric matrix set and $\mathcal{SA}_P^{n\times n}$ is trivial due to the fact that $\mathcal{S}^{n\times n} \cap \mathcal{A}^{n\times n} = \{0\}$.

The symmetric orthogonal (anti-)symmetric matrices were initially considered by Zhou, Hu and Zhang, associated with matrix equations and inverse eigenvalue problems, see [27]. Peng [17] has investigated the symmetric orthogonal symmetric solution to the matrix equation

$$A^T X A = B, \tag{1.1}$$

which arose in an inverse problem of structural modification or the dynamic behaviour of a structure [2, 3, 4, 7, 13, 14]. The symmetric skew anti-symmetric solution of (1.1) and its optimal approximation were also obtained in [18] by using the generalized singular value decomposition (GSVD). However, it may happen that the matrix equation (1.1) is inconsistent due to the inaccuracies in the measured data. In this case, we may consider the solution of (1.1) in the least-squares sense [6, 12, 21]. The purpose of this paper is to extend the results in [18] to the least-squares problem with a symmetric orthogonal anti-symmetric constraint, which can be described as follows:

Problem 1.1. *Given matrices $A \in \mathcal{R}^{n\times m}$, $B \in \mathcal{S}^{m\times m}$, $P \in \mathcal{SOR}^{n\times n}$ and $X^* \in \mathcal{S}^{n\times n}$. Let*

$$\mathcal{S}_E = \left\{ X \mid X \in \mathcal{SA}_P^{n\times n}, \|A^T X A - B\| = \min_{Y \in \mathcal{SA}_P^{n\times n}} \|A^T Y A - B\| \right\}. \tag{1.2}$$

Then find $\widehat{X} \in \mathcal{S}_E$ such that

$$\|\widehat{X} - X^*\| = \min_{X \in \mathcal{S}_E} \|X - X^*\|. \tag{1.3}$$

The minimization problem (1.3) arises in the structural modification and model updating [8]. The initial analytical matrix X^* is experimentally obtained from a practical measurement, but it may not satisfy the structural requirement or the minimum residual requirement. Hence, it is necessary to find the updated matrix $\widehat{X}$, which is not only a least-squares solution of matrix equation (1.1) with given structural requirement, but also a best approximation to the initial matrix X^*.

Similar to [12], the solution $\widehat{X}$ of Problem 1.1 can not be obtained by means of the canonical correlation decomposition (CCD) of a matrix pair, and the difficulty lies in the fact that the invariance of the Frobenius norm does not hold for general nonsingular matrices in CCD (see, for instance, (8) and (19) in [12]). In order to overcome this difficulty, a method, based on the projection theorem, GSVD and CCD, is adopted to solve Problem 1.1, and this approach has been applied successfully to find the least-squares solution of the matrix equations $(AXB, GXH) = (C, D)$ with minimum norm [15].

The outline of this paper is as follows. First, in Section 2, we will introduce several lemmas which will be used in the latter sections. Then, we will discuss Problem 1.1 and give the expression of its solution in Section 3. Finally, in Section 4, we will give the numerical algorithm to compute the solution of Problem 1.1 and report our numerical experiments.