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PRECONDITIONING HIGHER ORDER FINITE ELEMENT SYSTEMS BY ALGEBRAIC MULTIGRID METHOD OF LINEAR ELEMENTS *1)

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Abstract

We present and analyze a robust preconditioned conjugate gradient method for the higher order Lagrangian finite element systems of a class of elliptic problems. An auxiliary linear element stiffness matrix is chosen to be the preconditioner for higher order finite elements. Then an algebraic multigrid method of linear finite element is applied for solving the preconditioner. The optimal condition number which is independent of the mesh size is obtained. Numerical experiments confirm the efficiency of the algorithm.

Mathematics subject classification: 65N30, 65N55.
Key words: Finite element, Algebraic multigrid methods, Preconditioned Conjugate Gradient, Condition number.

1. Introduction

Multigrid method is one of the most efficient methods for solving large scale algebraic systems arising from the discretizations of partial differential equations(c.f. [1, 2, 9, 8, 10, 11]). The mesh size independent convergence rate can be achieved for geometric multigrid methods. For many practical problems, since the complexity of problems and solution domains, we have to use unstructured grids shown as in the Figure 2 and 3 for examples. The algebraic multigrid methods (AMG) are more suitable for the unstructured grids than geometric multigrid methods. A typical algebraic multigrid algorithm is like the algorithm 2.1, where the matrix B_h is the stiff matrix of the k order Lagrangian finite element. In the algebraic multigrid procedure, the coarsening of the grids is the most important issue but it is not easy to control the number of coarse grid degrees of freedom. The known AMG methods for finite element systems are designed mainly based on the linear element[3, 1]. Whether the convergence rate depends on mesh size or not is still open. The numerical examples show the dependence, see Table 2.1.

Lagrangian finite elements are important class of finite elements family in practical applications, which includes the linear and high order Lagrangian finite elements (see the Figure 1). The matrix structural of higher order finite element system is much complicated than the linear ones. The direct application of AMG algorithm for linear element to the higher order finite element yields the reduction of the efficiency (see the Table 2.1 and Table 2.2 for details). The more robust and efficient AMG algorithms need to be designed carefully.

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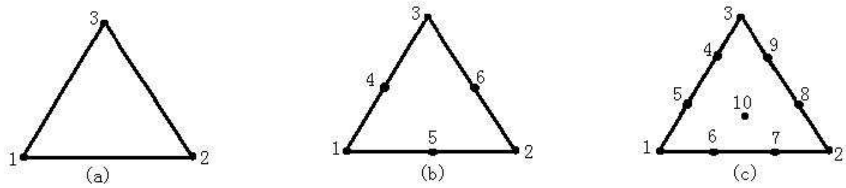

Figure 1: (a) The linear element. (b) The quadric element. (c) The cubic element.

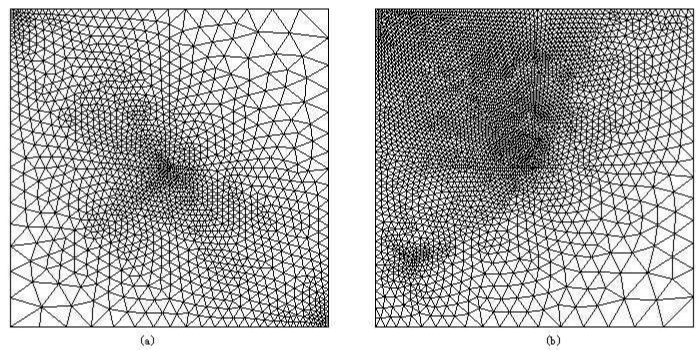

Figure 2: (a) The grid 1 with 2776 elements. (b) The grid 2 with 6427 elements.

Let T^h be a partition of Ω for a higher order Lagrangian finite element discretization, then we can introduce a refined grid T_h^l by refining the grid T^h through connecting the nodes, for example, the Figure 4 shows the grid T_h corresponding to quadric Lagrangian finite element and the refining grid T_h^l. Based on the new partition T_h^1 we can construct the stiffness matrix B_h of the linear Lagrangian finite element. This matrix B_h is supposed to be the preconditioner for the conjugate gradient algorithm for solving the discretization systems of high order Lagrangian finite element. The condition number of the preconditioned conjugate gradient methods is shown to be bounded independently on the mesh size. The numerical experiments confirm our theoretical results. The rigorous proof is given for the quadratic element and it can be extended to the higher order elements easily.

The rest of the paper is organized as follows. In section 2, we introduce the typical algebraic multigrid algorithm and give some comments. In section 3, for high order lagrangian finite elements, we give PCG methods based on algebraic multigrid method of linear finite element and provide some numerical experiments. Finally in section 4, we give a rigorous theoretical analysis for our PCG methods.

2. The Algebraic Multigrid Algorithm

For simplicity, we consider the following model problem

$$\begin{cases} -\nabla(a(x)\nabla u) = f, \quad x := (x_1, x_2) \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{2.1}$$

where $c_0 \le a(x) \le c_1$ and $c_0,\ c_1$ are positive constants.

Let T^h be the triangular partition of the domain Ω, and $\mathcal{P}_k$ be the set of polynomials of degree no more than k, where h is the maximal diameter of all the partition elements in T^h. We introduce the following Lagrangian finite element space.

Definition 2.1 $V_h^k = \{v_h^k(x) : v_h^k(x) \in C(\bar{\Omega}), v_h^k|_T \in \mathcal{P}_k, \forall T \in T^h\}$ *is called a* k *order La-*