Journal of Computational Mathematics, Vol.24, No.4, 2006, 481–500.

JACOBI PSEUDOSPECTRAL METHOD FOR FOURTH ORDER PROBLEMS *1)

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Abstract

In this paper, we investigate Jacobi pseudospectral method for fourth order problems. We establish some basic results on the Jacobi-Gauss-type interpolations in non-uniformly weighted Sobolev spaces, which serve as important tools in analysis of numerical quadratures, and numerical methods of differential and integral equations. Then we propose Jacobi pseudospectral schemes for several singular problems and multiple-dimensional problems of fourth order. Numerical results demonstrate the spectral accuracy of these schemes, and coincide well with theoretical analysis.

Mathematics subject classification: 65L60, 65M70, 41A05, 41A25.
Key words: Jacobi pseudospectral method, Differential equations of fourth order, Singular problems.

1. Introduction

The Jacobi polynomials play important roles in mathematical analysis and its applications. In particular, the Legendre and Chebyshev approximations have been used successfully for spectral and pseudospectral methods for non-singular differential equations, see [5, 9, 13, 14]. However, in many cases, we need to study other approximations. For instance, the usual Gauss-type interpolations are no longer available for numerical quadratures involving derivatives of functions at endpoints, and so we have to use certain specific Jacobi interpolations, see [12]. Next, in numerical analysis of finite element and boundary element methods, we took some results on the Jacobi approximations as important tools, see [1, 25, 26, 29]. The Jacobi approximations were also applied directly to numerical solutions of singular differential equations, and some problems on unbounded domains and aixymmetric domains, see [4, 15, 16, 17, 18]. Moreover, the Jacobi approximations are related to certain rational spectral methods and spectral method on triangle, see [11, 19, 20]. Recently, some authors developed the Jacobi spectral method for fourth order problems, see [23]. As for the Legendre spectral method for fourth order problems, we refer to the work of [6, 7, 27].

In actual computations, the pseudospectral method is more preferable, for which we only need to evaluate unknown functions at interpolation nodes, and thus save a lot of work. Especially, it is much easier to deal with various nonlinear problems. On the other hand, the most existing work are for second order problems. But it is also important to consider fourth order problems. For example, we may simulate incompressible fluid flows numerically, based

* Received April 30, 2005.
1) The work of these authors is supported in part by NSF of China, N.10471095, Science Foundation of Shanghai N.04JC14062, Special Funds for Doctorial Authorities of Chinese Education Ministry N.20040270002, Shanghai Leading Academic Discipline Project N.T0401, E-institutes of Shanghai Municipal Education Commission, N.E03004, Special Funds for Major Specialities and Fund N.04DB15 of Shanghai Education Commission.

on the stream function form of the Navier-Stokes equations, which fulfills the incompressibility automatically, and keeps physical boundary conditions. Since this is a nonlinear problem, we prefer to using pseudospectral method in actual computation.

The mathematical foundation of pseudospectral method for fourth order problems is the Jacobi-Gauss-type interpolations. In the early work, one considered these approximations in the standard Sobolev spaces, see [9, 14]. But, in many practical problems, the coefficients of derivatives of unknown functions involved in differential equations degenerate in different ways. Thus, the exact solutions are not in the standard Sobolev spaces. In other words, these problems are well- posed in certain non-uniformly weighted Sobolev spaces. Therefore, we have to investigate the Jacobi-Gauss-type interpolations in corresponding non-uniformly weighted Sobolev spaces. Some results on such approximations were established in [21, 22], which are very useful for pseudospectral method of second order problems. However, so far, there is no results which are appropriate for pseudospectral method of fourth order problems.

In this paper, we develop the Jacobi pseudospectral method for fourth order problems. We first establish some basic results on the Jacobi Gauss-type interpolations in certain non-uniformly weighted Sobolev spaces, which play important roles in the analysis of Jacobi pseudospectral method for fourth order problems. Then we propose the Jacobi pseudospectral schemes for several fourth order problems, such as singular differential equations and certain related problems. We also present some numerical results which demonstrate the spectral accuracy of proposed schemes, and coincide very well with theoretical analysis.

The paper is organized as follows. In the next section, we recall some recent results on the Jacobi polynomial approximation. In Section 3, we derive the basic results on the Jacobi-Gauss-type interpolations. In Section 4, we propose the Jacobi pseudospectral schemes for several problems of fourth order, and prove their convergence. In section 5, we present some numerical results. The final section is for concluding remarks.

2. Preliminaries

Let $\Lambda = \{x \mid |x| < 1\}$ and $\chi(x)$ be a certain weight function. Denote by $\mathbb{N}$ the set of all nonnegative integers. For any $r \in \mathbb{N}$, we define the weighted Sobolev space $H^r_\chi(\Lambda)$ as usual, with the inner product $(u, v)_{r,\chi}$, the semi-norm $|v|_{r,\chi}$ and the norm $\|v\|_{r,\chi}$. In particular, we denote by $(u, v)_\chi$ and $\|v\|_\chi$ the inner product and the norm of $L^2_\chi(\Lambda)$, respectively. For any $r > 0$, we define $H^r_\chi(\Lambda)$ and its norm by space interpolation as in [3]. The space $H^r_{0,\chi}(\Lambda)$ stands for the closure in $H^r_\chi(\Lambda)$ of the set $\mathcal{D}(\Lambda)$ consisting of all infinitely differentiable functions with compact support in Λ. When $\chi(x) \equiv 1$, we omit χ in the notations.

Denote by $J^{(\alpha,\beta)}_l(x), l = 1, 2, \dots$ the Jacobi polynomials. Let $\chi^{(\alpha,\beta)}(x) = (1 - x)^\alpha(1 + x)^\beta,\ \alpha, \beta > -1$. The set of Jacobi polynomials is the $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$-orthogonal system.

For any $N \in \mathbb{N}$, $\mathcal{P}_N$ denotes the set of all algebraic polynomials of degree at most N. The orthogonal projection $P_{N,\alpha,\beta} : L^2_{\chi^{(\alpha,\beta)}}(\Lambda) \to \mathcal{P}_N$ is defined by

$$(P_{N,\alpha,\beta}v - v, \phi)_{\chi^{(\alpha,\beta)}} = 0, \quad \forall \phi \in \mathcal{P}_N.$$

In order to describe the approximation errors, we introduce the space $H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$. For $r \in \mathbb{N}$, its semi-norm and norm are given by

$$|v|_{r,\chi^{(\alpha,\beta)},A} = \|\partial^r_x v\|_{\chi^{(\alpha+r,\beta+r)}}, \qquad \|v\|_{r,\chi^{(\alpha,\beta)},A} = (\sum_{k=0}^{r} \|\partial^k_x v\|^2_{\chi^{(\alpha+k,\beta+k)}})^{\frac{1}{2}}.$$

For any $r > 0$, we define the space and its norm by space interpolation as in [3].

Due to Theorem 2.1 of [22], we have that for any $v \in H^r_{\chi^{(\alpha,\beta)},A}(\Lambda), r \in \mathbb{N}$ and $0 \le \mu \le r$,

$$\|P_{N,\alpha,\beta}v - v\|_{\mu,\chi^{(\alpha,\beta)},A} \le cN^{\mu-r}|v|_{r,\chi^{(\alpha,\beta)},A}. \tag{2.1}$$