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ANTI-DIFFUSIVE FINITE DIFFERENCE WENO METHODS FOR SHALLOW WATER WITH TRANSPORT OF POLLUTANT *1)

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Dedicated to the 70th birthday of Professor Lin Qun

Abstract

In this paper we further explore and apply our recent anti-diffusive flux corrected high order finite difference WENO schemes for conservation laws [18] to compute the Saint-Venant system of shallow water equations with pollutant propagation, which is described by a transport equation. The motivation is that the high order anti-diffusive WENO scheme for conservation laws produces sharp resolution of contact discontinuities while keeping high order accuracy for the approximation in the smooth region of the solution. The application of the anti-diffusive high order WENO scheme to the Saint-Venant system of shallow water equations with transport of pollutant achieves high resolution

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1. Introduction

In this paper, we are interested in computing the transport of a passive pollutant in the flow modeled by the Saint-Venant system, given in the one dimensional case by

$$\begin{cases} h_t + (hu)_x = S \\ (hu)_t + (hu^2 + \frac{gh^2}{2})_x = -ghB_x \end{cases} \tag{1.1}$$

which is introduced in [16] and regularly used as a simplified model to describe shallow water flows. Here h is the depth, u is the velocity of water, g is the gravity constant, S is the pollutant source term, and $B(x)$ is the bottom topography. We are interested in locating the exact position and the correct concentration of the pollutant which is decided by a transport equation

$$(hT)_t + (uhT)_x = T_s S \tag{1.2}$$

where T is the pollutant concentration, and T_s is the concentration of the pollutant at the source. This model is used for the computation in [3] with a finite-volume particle (FVP) method. The FVP method is a hybrid method as a combination of two methods. For the

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shallow water equation (1.1), the finite volume method is used, and for the transport equation (1.2), the particle method is deployed. In [3], the authors also applied filters on the FVP method to smooth out the oscillations introduced by a combination of two different mechanisms.

The equation (1.2), which describes the transport of pollutant, is a linear equation for the variable hT for a given velocity u, thus the solution involving the pollutant will contain a contact discontinuity when initially hT is discontinuous. To locate the exact location and concentration of the pollutant, we need to resolve well the contact discontinuity in the solution, which is a difficult task as contact discontinuities, unlike shocks, are easily smeared by a shock capturing numerical method. There have been a lot of efforts in the literature to overcome the problem of the smearing of contact discontinuities. We refer, e.g., to [5, 6, 19] and the references therein.

Recently, Després and Lagoutière [4] proposed a new approach called limited downwind scheme, much akin to a class of flux limiters by Sweby [17], to prevent the smearing of contact discontinuities while keeping nonlinear stability. Their scheme is identical with the Superbee scheme developed by Roe [11] in the case of linear advection. By introducing an anti-diffusive flux, it gives remarkably sharp profiles of contact discontinuities in both one dimensional scalar and system cases. More importantly, they observe numerically and prove theoretically that their scheme adopts a class of *moving* traveling wave solutions exactly. This has an important implication that the smearing of contact discontinuities will not be progressively more severe for longer time, but will be stabilized for all time. A later paper by Bouchut [1] further modifies this scheme to satisfy entropy conditions and also gives a simple explicit formula for this limited downwind anti-diffusive flux.

In [18], we generalized the downwind flux correction idea to two dimensions and we developed a class of anti-diffusive high order finite difference WENO schemes to resolve contact discontinuities for conservation law equations. By going to high order accuracy, we were able to remove the unpleasant stairs in smooth regions when a first order anti-diffusive scheme is used. Ample numerical results in [18] indicate that our scheme can resolve well the contact discontinuities and at the same time maintains the stability and accuracy of regular high order WENO schemes for shocks and smooth structures of the solution. In this paper, we would like to further explore and apply the high order anti-diffusive finite difference WENO schemes in [18] to solve the equations (1.1) and (1.2) as a system, with the objective of obtaining sharp resolution of the contact discontinuities of the pollutant propagation.

High order finite difference WENO schemes in [9] were developed based on the successful ENO schemes [7, 14, 15] and third order finite volume WENO schemes [10], and have been quite successful in computational fluid dynamics and other applications. They are especially suitable for problems containing both shocks and complicated smooth flow features. For more details, we refer to the lecture notes [12] and the survey paper [13], and the references therein.

This paper is organized as follows. In Section 2 we briefly review the anti-diffusive high order finite difference WENO schemes in [18] with improved non-smoothness indicators. In Section 3 we apply this anti-diffusive finite difference scheme on the system (1.1) and (1.2) and give numerical results. In Section 4 we apply the anti-diffusive finite difference scheme on two dimensional models and show the success of the application through typical numerical tests. Concluding remarks are given in Section 5.

2. Flux Corrections for High Order Finite Difference WENO Schemes for Conservation Laws

In this section, we briefly review the techniques developed and applied in [18] for the conservation law equation

$$u_t + f(u)_x = 0 \tag{2.1}$$

with the assumption $f'(u) > 0$, for simplicity. The scheme for the other case $f'(u) < 0$ can be