Journal of Computational Mathematics, Vol.24, No.1, 2006, 91–102.

THE EFFECT OF MEMORY TERMS IN DIFFUSION PHENOMENA*1)

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Abstract

In this paper the effect of integral memory terms in the behavior of diffusion phenomena is studied. The energy functional associated with different models is analyzed and stability inequalities are established. Approximation methods for the computation of the solution of the integro-differential equations are constructed. Numerical results are included.

Mathematics subject classifications: 74D99,74S20, 78M20, 80M20.
Key words: Heat propagation, Integro-differential equation, Numerical approximation, Splitting method.

1. Heat Equation and Jeffrey's Kernel

Let us consider the problem of heat conduction in a one dimensional homogeneous and isotropic bar $(0,a)$ in which the heat pulses are transmitted by waves at finite but perhaps high speed. Representing by $q(x,t)$ the heat flux and assuming that holds the Fourier law

$$q = -k_1 \frac{\partial u}{\partial x}\,, \tag{1}$$

where k_1 is the effective thermal conductivity, it can be shown that the temperature u at (x,t) satisfies the classical heat equation

$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2}\,, \tag{2}$$

where c represents the thermal diffusivity. It is well known that this equation has the unphysical property that if a sudden change in the temperature is made at some point of the bar, it will be felt instantly everywhere. We say that diffusion gives rise to infinite speeds of propagation.

The problem that unphysical infinite speeds of propagation are generated by diffusion was first treated in [3]. In order to avoid this serious drawback it has been proposed in [3] to define the flux by an integral over the history of the temperature gradient, that is,

$$q(x,t) = -\frac{k}{\tau}\int_{-\infty}^{t} e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(x,s)\,ds\ , \tag{3}$$

where k represents the thermal conductivity. We note that the Fourier law holds as the limit of Cattaneo's law (3) when $\tau \to 0$. This definition of $q(x,t)$ corresponds to a first order approximation, in τ, of the modified Fourier law

$$q(x,t+\tau) = -k_1\frac{\partial u}{\partial x}(x,t).$$

In fact, considering the first order approximation

$$q(x,t+\tau) \simeq q(x,t) + \tau\frac{\partial q}{\partial t}(x,t),$$

* Received May 16, 2005.
1) This work has been supported by Centro de Matemática da Universidade de Coimbra and POCTI/35039/MAT/2000.

and integrating the first order differential equation

$$\frac{1}{\tau}q(x,t) + \frac{\partial q}{\partial t}(x,t) = -\frac{k_1}{\tau}\frac{\partial u}{\partial x}(x,t),$$

we obtain (3).

Considering (3), it can be shown that the temperature u at (x,t) satisfies Cattaneo's equation

$$\frac{\partial u}{\partial t}(x,t) = \frac{k}{\gamma\tau}\int_{-\infty}^{t} e^{-\frac{t-s}{\tau}}\frac{\partial^2 u}{\partial x^2}(x,s)\, ds\,, \tag{4}$$

where γ is the heat capacity. This equation was considered by different authors. For instance, Vernotte, in [15], considered Cattaneo's equation as the simplest that gives rise to finite speed of propagation. In fact, equation (4) is equivalent to the hyperbolic telegraph equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{1}{\tau}\frac{\partial u}{\partial t} = \frac{k}{\gamma\tau}\frac{\partial^2 u}{\partial t^2}\,, \tag{5}$$

which transmits waves with a finite speed $\sqrt{\dfrac{k}{\gamma\tau}}$ and presents a very small attenuation as a consequence of relaxation. The telegraph equation is the simplest mathematical model combining wave propagation and diffusion.

In Figure 1 we show the long time behavior of heat equation and Cattaneo's equation. The plots have been obtained from the discretization with standard numerical methods in a very fine mesh.

However, as pointed out in the engineering literature (see for example [9]), there are no real conductors which exhibit the wave propagation behavior of Cattaneo's model.

In [9] a corrected version of flux (3) is presented. A kernel of Jeffrey's type was then considered by replacing in (3) the exponential kernel by

$$Q(s) = k_1\delta(s) + \frac{k_2}{\tau}e^{-\frac{s}{\tau}}, \tag{6}$$

where $\delta(s)$ is a Dirac delta function, and k_1 and k_2 represent , respectively, the effective thermal conductivity and the elastic conductivity. In this case the Fourier law leads to a flux q defined by

$$q(x,t) = -k_1\frac{\partial u}{\partial x}(x,t) - \frac{k_2}{\tau}\int_{-\infty}^{t} e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(x,s)\, ds. \tag{7}$$

It can be shown that the temperature, in this case, satisfies Jeffrey's integro-differential equation

$$\frac{\partial u}{\partial t}(x,t) = \frac{k_1}{\gamma}\frac{\partial^2 u}{\partial x^2}(x,t) + \frac{k_2}{\gamma\tau}\int_{-\infty}^{t} e^{-\frac{t-s}{\tau}}\frac{\partial^2 u}{\partial x^2}(x,s)\, ds\,. \tag{8}$$

In recent years several authors gave attention to the introduction of Volterra integrals to model heat propagation (see [2], [5], [13]).

For $k_2 = 0$ we have the classical diffusion equation while for $k_1 = 0$ we obtain Cattaneo's equation. In Figure 2 we present the behavior of the three models at different times. We remark that Jeffrey's model allows the selection of parameters k_1 and k_2 such that mathematical models in agreement with experimental behavior of different materials can be obtained.

Cattaneo's equation and Jeffrey's equations predict different quantitative and qualitative behavior for the propagation of heat. This fact can be explained because while Cattaneo's equation is of hyperbolic type, Jeffrey's equation has a parabolic behavior. In the first case, if the initial condition presents discontinuities they will be propagated with constant speed. By the contrary, as Jeffrey's equation is of parabolic type, any discontinuity of the initial condition will be smoothed by diffusion associated with the effective thermal conductivity.