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REMARKS ON ERROR ESTIMATES FOR THE TRUNC PLATE ELEMENT *1)

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Abstract

This paper provides a simplified derivation for error estimates of the TRUNC plate element. The error analysis for the problem with mixed boundary conditions is also discussed.

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1. Introduction

The TRUNC element is very effective for the numerical solution of Kirchhoff plates. Applications to some sample problems showed that it converged rapidly [1, 2, 3]. Shi first established the error estimates in [9], and the derivation is rather technical.

This paper intends to revisit error analysis of the element. We will give a simple but very useful identity for the approximate solution. From this identity, we obtain a desired estimate for the term $E_1(u^*, \bar{w}_h)$ in [9] in a simplified way, which is essential in producing optimal error estimates. We also discuss error analysis of the method for corresponding problems with mixed boundary conditions. It deserves to point out that our derivation is different from that in [14], where the deduction of (3.18) is not rigorous (see Remark 1.4.4.7 in [6, p.32]).

2. Error Estimates for Plate Bending Problem with Clamped Conditions

Given a polygonal domain Ω, consider the following plate bending problem with clamped conditions [5]:

$$\begin{cases} -\mathcal{M}_{\alpha\beta,\alpha\beta}(u^*) = \Delta^2 u^* = f \text{ in } \Omega, \\ u^* = \partial_n u^* = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

where

$$\mathcal{M}_{\alpha\beta}(u) := (1-\sigma)\mathcal{K}_{\alpha\beta}(u) + \sigma\mathcal{K}_{\mu\mu}(u),\ \mathcal{K}_{\alpha\beta}(u) := -\partial_{\alpha\beta}u,\ 1 \le \alpha,\, \mu,\, \beta \le 2,$$

with $\sigma \in (0, 0.5)$ being the Poisson ratio of the plate and n the unit outward normal to $\partial\Omega$. Throughout this paper we use Einstein's convention for summation, and always assume that

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$u^* \in H^3(\Omega) \cap H_0^2(\Omega)$ in this section. The variational formulation of (2.1) is to find $u^* \in V = H_0^2(\Omega)$ such that

$$a(u^*, v) = f(v) = \int_\Omega f v dx, \ \ \forall v \in V,$$

where

$$\begin{aligned} a(u,v) :=& \int_\Omega \mathcal{M}_{\alpha\beta}(u)\mathcal{K}_{\alpha\beta}(v)dx \\ =& \int_\Omega [\Delta u \Delta v + (1-\sigma)(2\partial_{12}u\partial_{12}v - \partial_{11}u\partial_{22}v - \partial_{22}u\partial_{11}v)]dx. \end{aligned}$$

We next give some useful identities [8] for later uses. Given a polygon G, let v be a function in $H^3(G)$ and w a function in $H^2(G)$. Then

$$\begin{aligned} a_G(v,w) :=& \int_G \mathcal{M}_{\alpha\beta}(v)\mathcal{K}_{\alpha\beta}(w)dx \\ =& \int_G \mathcal{Q}_\alpha(v)\partial_\alpha w dx - \int_{\partial G}\{\mathcal{M}_{nn}(v)\partial_n w + \mathcal{M}_{n\tau}(v)\partial_\tau w\}ds, \end{aligned} \tag{2.2}$$

where

$$\mathcal{M}_{nn}(v) := \mathcal{M}_{\alpha\beta}(v)n_\alpha n_\beta, \ \mathcal{M}_{n\tau}(v) := \mathcal{M}_{\alpha\beta}(v)n_\alpha\tau_\beta, \ \mathcal{Q}_\alpha(v) := \partial_\beta\mathcal{M}_{\alpha\beta}(v),$$

with $n = (n_1, n_2)$ and $\tau = (\tau_1, \tau_2)$ being the unit outward normal and tangent vector to ∂G such that (n, τ) forms a right-hand system. Moreover, we have by (2.1) that

$$\int_G \mathcal{Q}_\alpha(u)\partial_\alpha v dx - f(v) = \int_{\partial G}\mathcal{Q}_n(u)v ds, \ \forall v \in H^1(G), \tag{2.3}$$

where $\mathcal{Q}_n(u) := \mathcal{Q}_\alpha(u)n_\alpha \in H^{-1/2}(\partial G)$. Since the tangent derivative is only the derivative with respect to the arc length parameter s in the boundary ∂G, we also write ∂_s for ∂_τ in what follows.

We divide the region of interest Ω into a regular family of triangular elements K with the diameter $h_K \le h$, $\bar{\Omega} = \cup_{K\in\mathcal{T}_h}\bar{K}$, and define on each triangle K the shape function to be an incomplete cubic polynomial,

$$\begin{aligned} v_h =& a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_1\lambda_2 + a_5\lambda_2\lambda_3 + a_6\lambda_3\lambda_1 \\ &+ a_7(\lambda_1^2\lambda_2 - \lambda_1\lambda_2^2) + a_8(\lambda_2^2\lambda_3 - \lambda_2\lambda_3^2) + a_9(\lambda_3^2\lambda_1 - \lambda_3\lambda_1^2), \end{aligned} \tag{2.4}$$

with the nodal parameters being the function values and the values of two first order derivatives at vertices of the triangle K, i.e., $v_h(p_i)$, $\partial_1 v_h(p_i)$, $\partial_2 v_h(p_i)$, $1 \le i \le 3$, where $\{p_i\}_{i=1}^3$ denote the three vertices of K. We then obtain the usual Zienkiewicz element space V_h related to V.

For each $v_h \in V_h$, we split the function into two parts,

$$v_h := \bar{v}_h + v_h', \tag{2.5}$$

where

$$\bar{v}_h|_K := a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_1\lambda_2 + a_5\lambda_2\lambda_3 + a_6\lambda_3\lambda_1 \tag{2.6}$$

and

$$v_h'|_K := a_7(\lambda_1^2\lambda_2 - \lambda_1\lambda_2^2) + a_8(\lambda_2^2\lambda_3 - \lambda_2\lambda_3^2) + a_9(\lambda_3^2\lambda_1 - \lambda_3\lambda_1^2). \tag{2.7}$$

Thus, we define a bilinear form on V_h by

$$b_h(u_h, v_h) := a_h(\bar{u}_h, \bar{v}_h) + a_h(u_h', v_h'), \ \forall u_h, \ v_h \in V_h,$$