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A DYNAMICS APPROACH TO THE COMPUTATION OF EIGENVECTORS OF MATRICES *

S. Jiménez
(*Colaborador honorífico, Dept. Matemática Aplicada, Universidad Complutense de Madrid, 28040 Madrid, Spain*)

L. Vázquez
(*Dept. Matemática Aplicada, Facultad de Informática Universidad Complutense de Madrid, 28040 Madrid, Spain; Centro de Astrobiología, CSIC/INTA, 28850 Torrejón de Ardoz, Madrid, Spain*)

Abstract

We construct a family of dynamical systems whose evolution converges to the eigenvectors of a general square matrix, not necessarily symmetric. We analyze the convergence of those systems and perform numerical tests. Some examples and comparisons with the power methods are presented.

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1. Introduction

In a previous work [1, 2], a method was proposed to solve systems of linear equations $A\vec{x} = \vec{b}$, by means of considering a dissipative mechanical system associated to the matrix A. This mechanical system evolves under Newton's Second Law towards the solution of the linear system. A numerical simulation was proposed then to calculate the solution in an iterative procedure.

Following a similar point of view, in this paper we construct dynamical systems that have as critical points the eigenvectors of a real square matrix and that evolve towards an eigenvector. In section 2 we present the dynamical systems and their basic properties. In section 3 a numerical scheme is proposed to simulate the evolution and some examples and applications are presented. The main conclusions are summarized in Section 4. Finally, we present the proof of the results in an Appendix.

2. The Dynamical Systems

Let us consider the dynamical system

$$\dot{\vec{x}} = -\frac{A}{\|\vec{x}\|^p}\,\vec{x} + \frac{\vec{x}^{\mathrm{T}} A\vec{x}}{\|\vec{x}\|^{p+2}}\,\vec{x} \tag{1}$$

where $p \in \mathbb{R}$, $\vec{x} \in \mathbb{R}^q$ for some $q \in \mathbb{N}$ and A is a real, $q \times q$ matrix. The norm $\|\ \|$ is the euclidean vectorial norm. If A is symmetric and $p = 2$, the system is equivalent to

$$\dot{\vec{x}} = -\vec{\nabla} U(\vec{x}), \quad U(\vec{x}) = \frac{1}{2}\,\frac{\vec{x}^{\mathrm{T}} A\vec{x}}{\|\vec{x}\|^2}, \tag{2}$$

but we are not assuming any restriction on A.

This system has the following basic properties:

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1. The fixed points are the eigenvectors of A (and conversely).

2. Conservation law:
$$\frac{\mathrm{d}\|\vec{x}\|^2}{\mathrm{d}t} = \vec{0}. \tag{3}$$

3. Only if A es symmetric and $p = 2$, dissipation law:
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[U(\vec{x})\right] = -\dot{\vec{x}}^2. \tag{4}$$

They are established in a straightforward way: in the first case the equivalence is clear. The conservation and the dissipation laws are obtained taking scalar product with $\vec{x}$ in (**??**) and $\dot{\vec{x}}$ in (2), respectively.

The conservation law (3) supposes that given an initial data $\vec{x}_0$, the corresponding solution lies for all times on the sphere $\|\vec{x}(t)\| = \|\vec{x}_0\|$. By Chillingworth's Theorem (see, for instance, Theorem 1.0.3 in [3]), we know that the solution always exists and is unique provided $\vec{x}_0 \neq \vec{0}$.

By direct calculation, the jacobian $J_p(\vec{x})$ of the dynamical system (**??**) at a given vector $\vec{x}$ is:
$$J_p(\vec{x}) = \frac{1}{\|\vec{x}\|^p}\left(\Big[A - r(\vec{x})I\Big]\Big[pP(\vec{x}) - I\Big] + P(\vec{x})\left[A + A^{\mathrm{T}} - 2r(\vec{x})I\right]\right). \tag{5}$$

Here I stands for the $q \times q$ identity matrix, $P(\vec{x})$ is the orthogonal projector on $\mathrm{span}\{\vec{x}\}$ and $r(\vec{x})$ is the Rayleigh quotient at $\vec{x}$:
$$P(\vec{x}) = \frac{\vec{x}\,\vec{x}^{+}}{\|\vec{x}\|^2}, \qquad r(\vec{x}) = \frac{\vec{x}^{+}A\vec{x}}{\|\vec{x}\|^2},$$
where the superscript $^{+}$ denotes the transposed, complex conjugate (or in case the vector is real, just the transposed).

Let us consider now the linear stability of the critical points. Let $\vec{u}$ be an eigenvector of A associated to the eigenvalue λ. From Lemma 1 in the Appendix, we have that $J_p(\vec{u})$ is a singular matrix and that the very eigenvector $\vec{u}$ belongs to the kernel:
$$J_p(\vec{u})\vec{u} = \frac{-1}{\|\vec{u}\|^p}\Big[I - P(\vec{u})\Big][A - \lambda I]\vec{u} = \vec{0}. \tag{6}$$

Thus, we see that the jacobian at any critical point has at least one eigenvector with zero real part. This means that in principle nothing can be said on the stability of the critical points from the study of the linear part. If we consider a symmetric A, the conservation law would allow us to conclude that the eigensubspace associated to the smallest eigenvalue is asymptotically stable. In the general case, we have to consider that given an initial data $\vec{x}_0$, the evolution is confined to the surface of the sphere $\|\vec{x}(t)\| = \|\vec{x}_0\|$, thus in order to check linear stability we must restrict ourselves to this manifold. The normal direction to the surface at $\vec{u}$ is given precisely by $\vec{u}$, thus we need to know the local behaviour around $\vec{u}$ in the orthogonal directions to $\vec{u}$. To do this, we compute all the eigenvalues of the jacobian, using the following result:

Theorem 1. *Let $\vec{u}$ be an eigenvector of A associated to the eigenvalue λ. The spectrum of A and that of $J_p(\vec{u})$ are related in the following way:*

1. *eigenvalue λ for A corresponds to eigenvalue 0 for $J_p(\vec{u})$*

2. *eigenvector $\vec{w}$ associated to μ for A corresponds to*
$$\textit{eigenvector} \quad \Big[I - P(\vec{u})\Big]\vec{w} \quad \textit{with eigenvalue} \quad \frac{\lambda - \mu}{\|\vec{u}\|^p} \quad \textit{for } J_p(\vec{u})$$
and this includes the case where $\mu = \lambda$, the case of complex eigenvalues and eigenvectors, as well as the case of generalized eigenvectors.