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COMPUTE A CELIS-DENNIS-TAPIA STEP *1)

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Abstract

In this paper, we present an algorithm for the CDT subproblem. This problem stems from computing a trust region step of an algorithm, which was first proposed by Celis, Dennis and Tapia for equality constrained optimization. Our algorithm considers general case of the CDT subproblem, convergence of the algorithm is proved. Numerical examples are also provided.

Mathematics subject classification: 65K10, 90C30.
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1. Introduction

The CDT subproblem has the following form,

$$\min_{d\in R^n} \Phi(d) = \frac{1}{2}d^T Bd + g^T d \tag{1.1}$$

$$\text{s.t.} \quad \|d\| \le \Delta,\ \|A^T d + c\| \le \xi, \tag{1.2}$$

where $g \in R^n$, $A \in R^{n\times m}$, $c \in R^m$, $\Delta > 0$, $\xi \ge 0$, and $B \in R^{n\times n}$ is a symmetric matrix. Throughout this paper, the norm $||\cdot||$ is the Euclidean norm. For convenience, we denote by $\mathcal{F}$ the feasible region of the CDT subproblem, namely,

$$\mathcal{F} = \{d \,|\, ||d|| \le \Delta, ||A^T d + c|| \le \xi\}. \tag{1.3}$$

As an important application, the CDT subproblem is a subproblem of some trust region algorithms for nonlinear programming, which was given by Celis, Dennis & Tapia[1] and Powell & Yuan[10], whose superlinear convergence property is obtained under certain conditions.

The properties of the CDT subproblem have been studied by many researchers, see Yuan[12], Peng & Yuan[9] and Chen & Yuan[2, 3, 4] etc. With some additional assumptions, some algorithms have been presented. For example, under the assumption that B is positive definite, two different algorithms have been proposed by Yuan[13] and Zhang[14] respectively. In this paper, we present an algorithm for solving problem (1.1)–(1.2) for general symmetric matrix B. We also assume that $\mathcal{F}$ has strict interior points.

The paper is organized as follows. In the next section, we state some known results which we will use in this paper. In section 3, we consider dual function and give some useful results.

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In section 4, the algorithm is presented. In section 5, convergence properties are analyzed. And in the last two section, we give numerical experiments.

2. Some Basic Results

In this section, we restate some fundamental results of the CDT subproblem.

Firstly, we define some notations as follows,

$$\mathcal{R}_+^2 = \{(\lambda,\mu) \mid \lambda \geq 0,\ \mu \geq 0\}, \tag{2.1}$$

$$\Omega_0 = \{(\lambda,\mu) \in \mathcal{R}_+^2 \mid H(\lambda,\mu) \text{ is positive semi-definite}\}, \tag{2.2}$$

$$\Omega_1 = \{(\lambda,\mu) \in \mathcal{R}_+^2 \mid H(\lambda,\mu) \text{ has one negative eigenvalue}\}, \tag{2.3}$$

where $H(\lambda,\mu) = B + \lambda I + \mu AA^T$. For the dual variables $\lambda \geq 0, \mu \geq 0$, when $g + \mu Ac \in \mathcal{R}(H(\lambda,\mu))$, we also define the vector

$$d(\lambda,\mu) = -H(\lambda,\mu)^+(g + \mu Ac), \tag{2.4}$$

and the Lagrangian dual function

$$\Psi(\lambda,\mu) = \Phi(d(\lambda,\mu)) + \frac{\lambda}{2}(||d(\lambda,\mu)||^2 - \Delta^2) + \frac{\mu}{2}(||A^T d(\lambda,\mu) + c||^2 - \xi^2). \tag{2.5}$$

The optimal conditions for the CDT subproblem were first proved by Yuan[12].

Theorem 2.1. *Let d^* be a global solution of the problem (1.1)–(1.2). Then there exist $\lambda^*, \mu^* \geq 0$ such that*

$$(B + \lambda^* I + \mu^* AA^T)d^* = -(g + \mu^* Ac), \tag{2.6}$$

$$\lambda^*(\Delta - ||d^*||) = 0, \quad \mu^*(\xi - ||A^T d^* + c||) = 0. \tag{2.7}$$

Furthermore, the matrix

$$H(\lambda^*,\mu^*) = B + \lambda^* I + \mu^* AA^T \tag{2.8}$$

has at most one negative eigenvalue if the multiplier λ^*, μ^* are unique.

In addition, Yuan[12] also showed that we could always find the global solution d^* with $(\lambda^*,\mu^*) \in \Omega_0 \cup \Omega_1$. From these results, we can try to get the global solution of the CDT subproblem by searching the dual variables (λ^*,μ^*) in $\Omega_0 \cup \Omega_1$. Further, Chen & Yuan[2] gave the criterion that the CDT subproblem has no global solution d^* with $(\lambda^*,\mu^*) \in \Omega_0$, which can be presented as follows.

Theorem 2.2. *If there is no global solution d^* of the CDT subproblem with the corresponding $H(\lambda^*,\mu^*)$ positive semi-definite, then the maxima (λ_+,μ_+) of dual function $\Psi(\lambda,\mu)$ in the region Ω_0 satisfy*

i) *$H(\lambda_+,\mu_+)$ is positive semi-definite with defect 1;*

ii)

$$(||d(\lambda_+,\mu_+) + \tilde{\tau}_+\tilde{u}|| - \Delta)(||d(\lambda_+,\mu_+) + \tilde{\tau}_-\tilde{u}|| - \Delta) < 0, \tag{2.9}$$

where $d(\lambda_+,\mu_+)$ is defined by (2.4), $\tilde{u}$ satisfies $||\tilde{u}|| = 1$ and $H(\lambda_+,\mu_+)\tilde{u} = 0$, and $\tilde{\tau}_\pm$ are respectively given by

$$\lambda_+(||d(\lambda_+,\mu_+) + \tilde{\tau}\tilde{u}||^2 - \Delta^2) + \mu_+(||A^T(d(\lambda_+,\mu_+) + \tilde{\tau}\tilde{u}) + c||^2 - \xi^2) = 0. \tag{2.10}$$