Journal of Computational Mathematics, Vol.23, No.4, 2005, 408–418.

ON HERMITIAN POSITIVE DEFINITE SOLUTIONS OF MATRIX EQUATION $X - A^*X^{-2}A = I$ *

Yu-hai Zhang
(*School of Mathematics and System Sciences, Shandong University, Jinan 250100, China*)

Abstract

The Hermitian positive definite solutions of the matrix equation $X - A^*X^{-2}A = I$ are studied. A theorem for existence of solutions is given for every complex matrix A. A solution in case A is normal is given. The basic fixed point iterations for the equation are discussed in detail. Some convergence conditions of the basic fixed point iterations to approximate the solutions to the equation are given.

Mathematics subject classification: 15A24, 65H05.
Key words: Matrix equation, Positive definite solution, Iterative methods.

1. Introduction

In this paper, we are concerned with the Hermitian positive definite solutions of the matrix equation

$$X - A^*X^{-2}A = I, \tag{1}$$

where I is the $n \times n$ identity matrix and A is an $n \times n$ complex matrix. The equation has been studied by several authors (see[6,8-11]) and some convergence conditions of the basic fixed point iterations to approximate the solutions to the equation are given. For the application areas in which the equation arises, see the references given in [6,8]. For the equation $X \pm A^*X^{-1}A = I$, there are many contributions in the literature on the theory, applications, and numerical solution(see,e.g.,[3-6,9-13]). Several authors[7,8,9,11,14] have studied the equation $X + A^*X^{-2}A = I$ and they have obtained theoretical properties of the equation.

Throughout this paper we use $\mathcal{C}^{n\times n}$ to denote the set of complex $n \times n$ matrices, and $\mathcal{H}^{n\times n}$ to denote the set of $n \times n$ Hermitian matrices. For $M \in \mathcal{C}^{n\times n}$, $\|M\|$ stands for the spectral norm and $\lambda_i(M)$ represents the eigenvalues. For $X, Y \in \mathcal{H}^{n\times n}$, we write $X \geq Y (X > Y)$ if $X - Y$ is positive semi-definite (definite). For $M \in \mathcal{H}^{n\times n}$, let $\lambda_{max}(M)$ and $\lambda_{min}(M)$ be maximal and minimal eigenvalue of M, respectively.

In Section 2 we discuss existence of solutions and their properties and consider the solutions in case A is normal or unitary. In Section 3 we give an estimation on the solutions. In Section 4, we discuss the convergence behavior of the basic fixed point iterations to approximate the solutions to Eq.(1). Some of results in [8,9,11] are improved. Several numerical examples are given in Section 5.

2. Existence of Solutions

In the sequel, a solution always means a Hermitian positive definite one.

Lemma 1. [9,10] *Eq.(1) has a solution for any* $A \in \mathcal{C}^{n\times n}$.

* Received April 13, 2004; final September 16, 2004.

Theorem 1. *For any invertible matrix $A \in \mathcal{C}^{n\times n}$, there exist unitary matrices P and Q and diagonal matrices $\Gamma > I$ and $\Sigma > 0$ with $\Gamma - \Sigma^2 = I$ such that*

$$A = P^*\Gamma Q\Sigma P.$$

In this case $X = P^*\Gamma P$ is a solution of Eq.(1).

Proof. For any $A \in \mathcal{C}^{n\times n}$, by Lemma 1 Eq.(1) has a solution. Suppose that X is a solution. Then there exist a unitary matrix P and a diagonal matrix Γ such that $X = P^*\Gamma P$. Hence, the identity $X = I + A^*X^{-2}A$ gives

$$\Gamma - I = PA^*P^*\Gamma^{-2}PAP^*.$$

Noticing that $\Gamma > I$, then we have

$$\left((\Gamma - I)^{-\frac{1}{2}}PA^*P^*\Gamma^{-1}\right)\left(\Gamma^{-1}PAP^*(\Gamma - I)^{-\frac{1}{2}}\right) = I.$$

Let $Q = \Gamma^{-1}PAP^*(\Gamma - I)^{-\frac{1}{2}}$, that is $A = P^*\Gamma Q\Sigma P$ with $\Sigma = (\Gamma - I)^{\frac{1}{2}}$. Obviously, Q is unitary and $\Gamma - \Sigma^2 = I$. It is easy to verify that $X = P^*\Gamma P$ is a solution of Eq.(1).

Theorem 2. *If A is normal, in other words, there exists a unitary matrix P such that $A = P^*\Lambda P$ where $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$, $\lambda_i,\ i = 1, 2, \cdots, n$ are the eigenvalues, then Eq.(1) has the following solution*

$$X = P^*diag(\mu_1, \mu_2, \cdots, \mu_n)P, \tag{2}$$

where μ_i is the unique positive solution of the equation

$$\mu_i - |\lambda_i|^2\mu_i^{-2} = 1 \tag{3}$$

for $i = 1, 2, \cdots, n$.

Proof. Let $Y = PXP^*$. Consequently, Eq.(1) has a solution if and only if the following problem is solvable:

$$\exists Y > 0,\ \ Y - \Lambda^*Y^{-2}\Lambda = I. \tag{4}$$

Note that the equation (3) has only one positive solution μ_i and $\mu_i \in [1, +\infty)$. Let $Y = diag(\mu_1, \mu_2, \cdots, \mu_n)$. It is easy to verify Y is a solution of (4).

Theorem 3. *If A is a unitary matrix, then Eq.(1) has only one solution $X = \delta I$, where δ is the unique positive solution of the following equation*

$$\delta = 1 + \delta^{-2}.$$

Proof. It is easy to prove that $X = \delta I$ is a solution of Eq.(1). Suppose that X is a solution of Eq.(1). We prove $X = \delta I$. We know that there exist a unitary matrix U and a diagonal matrix $\Delta = diag(\delta_1, \delta_2, \cdots, \delta_n)$ such that $X = U^*\Delta U$. Hence, the identity $X = I + A^*X^{-2}A$ gives

$$V(\Delta - I) = \Delta^{-2}V,$$

where $V = (v_{ij}) = UAU^*$. Obviously, V is unitary. Hence $detV \neq 0$, then there exists a permutation π of the n items $\{1, 2, \cdots, n\}$ such that $\prod_{i=1}^n v_{i,\pi(i)} \neq 0$. By computation, one derives for each i

$$v_{i,\pi(i)}(\delta_{\pi(i)} - 1) = \delta_i^{-2}v_{i,\pi(i)},$$

which implies

$$\delta_{\pi(i)} - 1 = \delta_i^{-2},\ \text{ for }\ i = 1, 2, \cdots, n.$$