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STABILITY OF THEORETICAL SOLUTION AND NUMERICAL SOLUTION OF NONLINEAR DIFFERENTIAL EQUATIONS WITH PIECEWISE DELAYS [*1)]

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Abstract

This paper is concerned with the stability of theoretical solution and numerical solution of a class of nonlinear differential equations with piecewise delays. At first, a sufficient condition for the stability of theoretical solution of these problems is given, then numerical stability and asymptotical stability are discussed for a class of multistep methods when applied to these problems.

Mathematics subject classification: 65L06, 65L20.
Key words: Stability, Delay differential equations, Linear multistep methods.

1. Introduction

In recent years, many authors discussed the stability of numerical methods for the solution of delay differential equations(DDEs) (see, e.g., [1, 2, 3, 4, 10] and their references) with constant delay. Recently, H.Tian [9] has given the exponential asymptotic stability of singularly perturbed delay differential equations with a bounded lag, and this type stability can be applied to general delay differential equations with a variable lag. However, the stability results of numerical methods for differential equations with variable delays are much less. In 1997, Zennaro[7] first investigated asymptotical stability of nonlinear delay differential equations(DDEs) with a variable delay, and gave the stability result of Runge-Kutta methods applied to this systems. In 1984, Cooke et al [6] described the existence, asymptotic behavior, periodic and oscillating solutions of the differential equations with piecewise constant delays. More results can also be found in [8] about differential equations with piecewise continuously variable arguments. In [5], the stability of θ-methods has been studied by Zhang Changhai et al, which is based on the linear problem

$$\begin{cases} y'(t) = ay(t) + by([t]), \quad t \geq 0, \\ \qquad y(0) = y_0, \end{cases}$$

where a, b denote real constants and $[\cdot]$ denotes the greatest integer function. In this paper, we further investigate the stability of the theoretical solution and numerical solution of a class of initial value problems in nonlinear differential equations with piecewise delays. In section 2, we fix our attention on the stability of the theoretical solution of the problems. In section 3, we analyze the stability and asymptotical stability of a class of linear multistep methods when applied to the problems. Our results are further verified by the numerical experiment in section 4.

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2. Test Problems

Let $\langle\cdot,\cdot\rangle$ be an inner product in C^N and $\|\cdot\|$ the corresponding norm. Consider the following initial value problem in nonlinear differential equations with piecewise delay:

$$\begin{cases} y'(t) = f(t, y(t), y([t])), \quad t \geq 0, \\ \qquad y(0) = y_0, \end{cases} \tag{2.1}$$

where $[\cdot]$ is the largest-integer function, and $f : [0, +\infty) \times C^N \times C^N \longrightarrow C^N$ is a given continuous mapping. Assume that there exist continuous bounded functions $\alpha(t)$ and $\beta(t)$ on the interval $[0, +\infty)$, which satisfies the following conditions:

$$\alpha(t) \leq 0, \quad \alpha(t) + \beta(t) \leq 0 \quad \forall t \geq 0, \tag{2.2}$$

such that

$$\begin{cases} Re < u_1 - u_2, f(t, u_1, v) - f(t, u_2, v) > \leq \alpha(t)\|u_1 - u_2\|^2, \forall t \geq 0, u_1, u_2, v \in C^N & (2.3a) \\ \|f(t, u, v_1) - f(t, u, v_2)\| \leq \beta(t)\|v_1 - v_2\|, \, \forall t \geq 0, u, v_1, v_2 \in C^N, & (2.3b) \end{cases}$$

and that the problem (2.1) has a unique true solution $y(t)$ on the interval $[0, +\infty)$.

In order to discuss the contractivity and asymptotic stability of (2.1), we introduce the perturbed problem

$$\begin{cases} z'(t) = f(t, z(t), z([t])), \quad t \geq 0, \\ \qquad z(0) = z_0, \end{cases} \tag{2.4}$$

and assume that the problem (2.4) has a unique true solution $z(t)$.

Theorem 2.1. *If the mapping f satisfies the condition* (2.3) *with (2.2), then we have*

$$\|y(t) - z(t)\| \leq \|y_0 - z_0\| \qquad \forall \;\; t \in [0, +\infty) \tag{2.5}$$

Proof. Define $Y(t) := \|y(t) - z(t)\|^2 =< y(t) - z(t), y(t) - z(t) >$. Noting the conditions (2.2) and (2.3), and Cauchy-Schwartz inequality, we have

$$\begin{aligned} Y'(t) &= 2Re < y(t) - z(t), y'(t) - z'(t) > \\ &= 2Re < y(t) - z(t), f(t, y(t), y([t])) - f(t, z(t), y([t])) > \\ &+ 2Re < y(t) - z(t), f(t, z(t), y([t])) - f(t, z(t), z([t])) > \\ &\leq 2\alpha(t)Y(t) + 2\beta(t)\|y(t) - z(t)\|\|y([t]) - z([t])\| \\ &\leq 2\alpha(t)Y(t) + \beta(t)(Y(t) + Y([t])) \\ &= \alpha(t)Y(t) + (\alpha(t) + \beta(t))Y(t) + \beta(t)Y([t]) \\ &\leq \alpha(t)Y(t) + \beta(t)Y([t]). \end{aligned}$$

Let $A(x) := \int_0^x \alpha(t)dt$, for every $t_0 \geq 0, t \geq t_0$, we have

$$\int_{t_0}^{t} (e^{-A(x)}Y(x))'dx \leq \int_{t_0}^{t} \beta(x)e^{-A(x)}Y([x])dx. \tag{2.6}$$

Hence

$$Y(t) \leq Y(t_0)e^{A(t)-A(t_0)} - e^{A(t)} \int_{t_0}^{t} \alpha(x)e^{-A(x)}Y([x])dx.$$

For the case $m \leq t \leq m + 1$ with integer $m \geq 0$. Let $t_0 = m$, we have

$$\begin{aligned} Y(t) &\leq Y(m)[e^{A(t)-A(m)} - e^{A(t)} \int_{m}^{t} \alpha(x)e^{-A(x)}dx] \\ &\leq Y(m)[e^{A(t)-A(m)} + 1 - e^{A(t)-A(m)}] \leq Y(m). \end{aligned} \tag{2.7}$$

By iterating, the (2.5) is true.

Modifying the conditions of theorem 2.1 further, we can obtain the following conclusion.