Journal of Computational Mathematics, Vol.23, No.3, 2005, 321–326.

CONVERGENCE OF INNER ITERATIONS SCHEME OF THE DISCRETE ORDINATE METHOD IN SPHERICAL GEOMETRY*1)

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Abstract

In transport theory, the convergence of the inner iteration scheme to the spherical neutron transport equation has been an open problem. In this paper, the inner iteration for a positive step function scheme is considered and its convergence in spherical geometry is proved.

Mathematics subject classification: 65N12, 65N22, 65F10.
Key words: Transport equation, Discrete ordinate, Inner iteration.

1. Introduction

In 1968, Carlson and Lathrop [1, p. 261] pointed out, there were some unsolved problems in neutron transport theory. One of them is the convergence of inner and outer iteration process. Perhaps investigations of these problems lead to procedures for accelerating convergence. Since then considerable progress has been made in convergence of inner iterations in slab geometry. For example, Menon and Sahni [2] estimated the spectral radius of the iteration matrix and proved the convergence theorem under the assumption of "non-regenerative" in slab geometry. Nelson [3] proved that the similar conclusion under the hypothesis of "weak non-multiplying". Recently Yuan et al[4] proved the convergence under a weaker condition on known data and more general boundary conditions.

Due to the appearance of the angular derivative in curvilinear geometry, the formalism of such inner iterations scheme are more complex and there is not any known convergent result by now. In this paper, we will establish the convergence of inner iterations to the spherical neutron transport equation. The means employed here is the Perron-Frobenius theory for non-negative matrix, but the argument method of Nelson's proof [3] is improved, just like that in [4] so as to handle complicated process. Although the result is concluded for a positive step scheme, it can be extended to some other positive schemes.

Consider the following neutron transport equation

$$\mu\frac{\partial\psi}{\partial r}+\frac{1-\mu^2}{r}\frac{\partial\psi}{\partial\mu}+\sigma(r)\psi=\frac{1}{2}c(r)\int_{-1}^{1}\psi(r,\mu')d\mu'+f(r), \tag{1}$$

where $r\in[0,R]$, $\mu\in[-1,1]$, ψ is angular flux with subject to vacuum boundary condition

$$\psi(R,\mu)=0,\quad \mu<0. \tag{2}$$

Here $\sigma(r)\geq c(r)\geq 0$ are the total and the scattering cross section respectively, $f(r)$ is the non-negative external source.

* Received December 10, 2003; final revised September 2, 2004.
1) Subsidized by the National Natural Science Foundation of China 10431050, 10476002, G19932010 and the Foundation of CAEP 20030658, 2003Z063, 20040653.

Eq.(1) is usually expressed the following conservative form too,

$$\frac{\mu}{r^2}\frac{\partial r^2\psi}{\partial r}+\frac{1}{r}\frac{\partial(1-\mu^2)\psi}{\partial\mu}+\sigma(r)\psi=\frac{1}{2}c(r)\int_{-1}^{1}\psi(r,\mu')d\mu'+f(r). \qquad (1')$$

Consider a spatial and angular net with mesh points $0=r_{\frac{1}{2}}<r_{\frac{3}{2}}<\cdots<r_{K+\frac{1}{2}}=R$; $-1=\mu_{\frac{1}{2}}<\mu_{\frac{3}{2}}<\cdots<\mu_{N+\frac{1}{2}}=1$, where N is an even number. Let $C_k=[r_{k-\frac{1}{2}},r_{k+\frac{1}{2}}]$ for $k=1,2,\cdots,K$. Suppose that in the interior of each C_k, $\sigma(r)$ and $c(r)$ are constants σ_k and c_k respectively, then the standard inner iteration schemes are as follows [5, pp. 230 or 9, pp. 141]:

$$\begin{aligned}&\mu_m(A_{k+\frac{1}{2}}\psi^{(n+1)}_{k+\frac{1}{2},m}-A_{k-\frac{1}{2}}\psi^{(n+1)}_{k-\frac{1}{2},m})\\&+(A_{k+\frac{1}{2}}-A_{k-\frac{1}{2}})\cdot\frac{\alpha_{m+\frac{1}{2}}\psi^{(n+1)}_{k,m+\frac{1}{2}}-\alpha_{m-\frac{1}{2}}\psi^{(n+1)}_{k,m-\frac{1}{2}}}{\omega_m}\\&+V_k\sigma_k\psi^{(n+1)}_{km}=S^{(n)}_k,\end{aligned} \qquad (3)$$

$$S^{(n)}_k=V_kc_k\sum_{j=1}^{N}\psi^{(n)}_{kj}\omega_j+V_kf_k.$$

$$m=1,2,\cdots,N;k=1,2,\cdots,K.$$

where $\mu_m\in[\mu_{m-\frac{1}{2}},\mu_{m+\frac{1}{2}}]$ are even order Gaussian quadrature sets, and ω_m are the corresponding weights, $\sum_{m=1}^{N}\omega_m=1$. $A_{k\pm\frac{1}{2}}=r^2_{k\pm\frac{1}{2}},V_k=\frac{1}{3}(r^3_{k+\frac{1}{2}}-r^3_{k-\frac{1}{2}})$, $\alpha_{m+\frac{1}{2}}-\alpha_{m-\frac{1}{2}}=-\mu_m\omega_m$ and $\alpha_{\frac{1}{2}}=0$.

Boundary conditions are

$$\psi^{(n+1)}_{K+\frac{1}{2},m}=0,\quad m=1,2,\cdots,\frac{N}{2}. \qquad (4)$$

The symmetry conditions at the center of the sphere are

$$\psi^{(n+1)}_{\frac{1}{2},N+1-m}=\psi^{(n+1)}_{\frac{1}{2},m},\quad m=1,2,\cdots,\frac{N}{2}. \qquad (5)$$

In order to complete the differencing procedure, we need two auxiliary relationships

$$\psi^{(n+1)}_{km}=a\psi^{(n+1)}_{k+\frac{1}{2},m}+(1-a)\psi^{(n+1)}_{k-\frac{1}{2},m}, \qquad (6)$$

$$\psi^{(n+1)}_{km}=b\psi^{(n+1)}_{k,m+\frac{1}{2}}+(1-b)\psi^{(n+1)}_{k,m-\frac{1}{2}}. \qquad (7)$$

where $0\le a\le1, 0\le b\le1, k=1,\cdots,K; m=1,\cdots,N$.

The starting direction equation ($\mu=-1$) is given by

$$-(A_{k+\frac{1}{2}}\psi^{(n+1)}_{k+\frac{1}{2},\frac{1}{2}}-A_{k-\frac{1}{2}}\psi^{(n+1)}_{k-\frac{1}{2},\frac{1}{2}})+V_k\sigma_k\psi^{(n+1)}_{k,\frac{1}{2}}=S^{(n)}_k. \qquad (8)$$

$$\psi^{(n+1)}_{k,\frac{1}{2}}=a\psi^{(n+1)}_{k+\frac{1}{2},\frac{1}{2}}+(1-a)\psi^{(n+1)}_{k-\frac{1}{2},\frac{1}{2}}, \qquad (9)$$

$$k=1,2,\cdots,K.$$

The boundary condition for the starting equation (8) is

$$\psi^{(n+1)}_{K+\frac{1}{2},\frac{1}{2}}=0. \qquad (10)$$