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AN ANISOTROPIC NONCONFORMING FINITE ELEMENT WITH SOME SUPERCONVERGENCE RESULTS *1)

Dong-yang Shi Shi-peng Mao Shao-chun Chen
(*Department of Mathematics, Zhengzhou University, Zhenzhou 450052, China*)

Abstract

The main aim of this paper is to study the error estimates of a nonconforming finite element with some superconvergence results under anisotropic meshes. The anisotropic interpolation error and consistency error estimates are obtained by using some novel approaches and techniques, respectively. Furthermore, the superclose and a superconvergence estimate on the central points of elements are also obtained without the regularity assumption and quasi-uniform assumption requirement on the meshes. Finally, a numerical test is carried out, which coincides with our theoretical analysis.

Mathematics subject classification: 65N30, 65N15.
Key words: Anisotropic meshes, Nonconforming finite element, Interpolation error and consistency error estimates, Superclose, Superconvergence.

1. Introduction

It is well-known that regular assumption or quasi-uniform assumption[1] of finite element meshes is a basic condition in analysis of finite element approximation both for conventional conforming and nonconforming elements. However, with the development of the finite element methods and its applications to more fields and more complex problems, the above regular assumption or quasi-uniform assumption are great deficient in the finite element methods. For example, the solution may have anisotropic behavior in parts of the domain. This means that the solution varies significantly only in certain directions. Such as the diffusion problems in domains with edges and singularly perturbed convection-diffusion-reaction problems where boundary or interior layers appear. In such cases, it is an obvious idea to reflect this anisotropy in the discretization by using anisotropic meshes with a small mesh size in the direction of the rapid variation of the solution and a larger mesh size in the perpendicular direction.

Considering a bounded convex domain $\Omega \subset R^2$, we can describe the elements of anisotropic meshes mathematically. Let J_h be a family of meshes of Ω and denote the diameter of the finite element K and the supremum of the diameters of all circles contained in K by h_K and ρ_K respectively, $h = \max\limits_{K \in J_h} h_K$. It is assumed in the classical finite element theory that $\frac{h_K}{\rho_K} \leq C$, where C be a positive constant which is independent of K and the function considered. Such assumption is no longer valid in the case of anisotropic meshes. Conversely, anisotropic elements K are characterized by $\frac{h_K}{\rho_K} \to \infty$, where the limit can be considered as $h \to 0$. Recently, Zenisek[2,3] and Apel[4,5] published a series of papers concentrating on the interpolation error estimates of some Lagrange Type elements(conforming elements), but nonconforming methods are hardly treated. As far as we know, it seems that there are few papers focused on the nonconforming elements under anisotropic meshes.

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On the other hand, the superconvergence study of the finite element methods is one of the most active research subject for a long time in theoretical analysis and practical computations. Many superconvergence results about conforming finite element methods have been obtained(see [6] [7]). Do these superconvergence results of conforming elements still hold for nonconforming ones? [8-10] studied the superconvergence of Wilson's element and obtained the superconvergence estimate of the gradient error on the centers of elements. Under square meshes, [11] recently obtained same superconvergentce results of rotated Q_1 element, too. However, to our knowledge, there are no papers published with respect to anisotropic meshes.

In our work, we firstly study the anisotropic interpolation property of a nonconforming finite element proposed by [12], which will play an important role in estimating the interpolation error. By employing some techniques different from the existing articles, we obtain the consistency error estimate. Then we get the superclose property and a superconvergence estimate on the centers of elements without the regularity assumption and quasi-uniform assumption requirements on the meshes. In the last section, some numerical examples are presented to illustrate the validity of our theoretical analysis.

2. Construction of the Finite Element Space with Anisotropic Interpolation Property

Assume $\hat{K} = [-1,1]\times[-1,1]$ to be the reference element, the four vertices are $\hat{d}_1 = (-1,-1), \hat{d}_2 = (1,-1), \hat{d}_3 = (1,1), \hat{d}_4 = (-1,1)$, let $\hat{l}_1 = \overline{\hat{d}_1\hat{d}_2}, \hat{l}_2 = \overline{\hat{d}_2\hat{d}_3}, \hat{l}_3 = \overline{\hat{d}_3\hat{d}_4}, \hat{l}_4 = \overline{\hat{d}_4\hat{d}_1}$.

We define the finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$ on $\hat{K}$ as follows

$$\hat{\Sigma} = \{\hat{v}_1, \hat{v}_2, \hat{v}_3, \hat{v}_4, \hat{v}_5\}, \qquad \hat{P} = span\{1, \xi, \eta, \varphi(\xi), \varphi(\eta)\}, \tag{1}$$

where $\hat{v}_i = \frac{1}{|\hat{l}_i|}\int_{\hat{l}_i} \hat{v} d\hat{s}, \quad i = 1,2,3,4, \quad \hat{v}_5 = \frac{1}{|\hat{K}|}\int_{\hat{K}} \hat{v} d\xi d\eta, \varphi(t) = \frac{1}{2}(3t^2-1)$.

It can be easily proved that the interpolation defined above is properly posed, the interpolation function is as follows

$$\hat{\Pi}\hat{v} = \hat{v}_5 + \frac{1}{2}(\hat{v}_2 - \hat{v}_4)\xi + \frac{1}{2}(\hat{v}_3 - \hat{v}_1)\eta + \frac{1}{2}(\hat{v}_2 + \hat{v}_4 - 2\hat{v}_5)\varphi(\xi) + \frac{1}{2}(\hat{v}_3 + \hat{v}_1 - 2\hat{v}_5)\varphi(\eta) \tag{2}$$

For the sake of convenience, Let $\Omega \subset R^2$ to be a convex polygon composed by a family of rectangular meshes J_h which doesn't need to satisfy the regularity conditions. $\forall K \in J_h$, denote the barycenter of element K by (x_K, y_K), the length of edges parallel to x-axis and y-axis by $2h_x, 2h_y$ respectively, $h_K = \max\{h_x, h_y\}, h = \max\limits_{K\in J_h} h_K$.

$F_K : \hat{K} \longrightarrow K$ is defined as

$$\begin{cases} x = x_K + h_x\xi, \\ y = y_K + h_y\eta. \end{cases} \tag{3}$$

Define the finite element space as

$$V_h = \{v_h | \hat{v_h} = v_h|_K \circ F_K \in \hat{P}, \ \forall K \in J_h, \int_F [v_h] ds = 0, F \subset \partial K\}, \tag{4}$$

where $[v_h]$ stands for the jump of v_h across the edge F if F is an internal edge, and it is equal to v_h itself if F is a boundary edge.

Let the general element K is a rectangle element in $x-y$ plane, the interpolate operator is defined as

$$\Pi_K : H^2(K) \to \hat{P} \circ F_K^{-1}, \Pi_K v = (\hat{\Pi}\hat{v}) \circ F_K^{-1}, \quad \Pi_h : H^2(\Omega) \to V_h, \Pi_h|_K = \Pi_K.$$

In order to obtain the anisotropic interpolation error estimate we should turn to the following lemma