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ON LOCKING-FREE FINITE ELEMENT SCHEMES FOR THREE-DIMENSIONAL ELASTICITY *1)

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Abstract

In the present paper, the authors discuss the locking phenomenon of the finite element method for three-dimensional elasticity as the Lamé constant $\lambda \to \infty$. Three kinds of finite elements are proposed and analyzed to approximate the three-dimensional elasticity with pure displacement boundary condition. Optimal order error estimates which are uniform with respect to $\lambda \in (0, +\infty)$ are obtained for three schemes. Furthermore, numerical results are presented to show that, our schemes are locking-free and and the trilinear conforming finite element scheme is locking.

Mathematics subject classification: 65N30, 73V05
Key words: Three-dimensional Elasticity, Locking-free, Nonconforming finite element.

1. Introduction

For the linear isotropic elasticity, it is well known that many numerical methods suffer deteriorations in performance as the Lame constant $\lambda \to \infty$, i.e., as the material becomes incompressible[1]. This is the so-called locking phenomenon. Many literatures concerning the planar elasticity have appeared to be locking-free[2] [3] [9] [13] [14] [15]. In 1983, M. Vogelius [18] considered conforming finite element approximations to the linear planar elasticity as $\lambda \to \infty$. He showed that the piecewise linear conforming finite element scheme did not converge any more. For higher order conforming finite element schemes, optimal error estimates could not be obtained. To overcome the locking, we need to construct some finite element schemes whose optimal error estimates are uniform with respect to $\lambda \in (0, \infty)$. They are nonconforming in general. In [2],[3],[14] and [15], some nonconforming finite finite elements are constructed and analyzed, to be locking-free. The authors obtained optimal error estimates uniform for $\lambda \in (0, \infty)$, by virtue of the variational formula of pure displacement boundary value problem, based on the minimization of the energy functional. The pure traction boundary value problem was considered in [9], [12] and [16] by triangular element approximations, in [21] by quadrilateral element approximations and [13] by the NRQ_1 element approximations following the argument of [21] by the mixed finite element analysis.

To the best of our knowledge, no paper deals with the locking phenomenon of three-dimensional elasticity by finite element methods. Since discrete variational formulas, based on the minimization of the energy functional, are easier to be solved than the mixed formula, we consider this formula with pure displacement boundary condition. In the present paper, the three-dimensional Crouzeix-Raviart element is showed to be locking-free and the optimal error estimate is obtain. We construct two kinds of nonconforming cuboidal finite elements showed to be locking-free and obtain optimal error estimates of them. The order of one of our schemes is the lowest. We also present some numerical experiments to show the locking phenomenon of

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the trilinear conforming finite element and the locking-free of our lowest order nonconforming finite element. The conforming element converges well when λ is small, but loses convergency when $\lambda \to \infty$. Our lowest locking-free scheme converges very well and uniformly for $\lambda \in (0, \infty)$.

The paper is arranged as follows: In section 2, we present the preliminary consideration of three-dimensional linear elasticity with pure displacement boundary condition, the locking phenomenon of conforming finite element method and the construction of locking-free finite element method. In section 3, we present the Crouziex-Raviart tetrahedral finite element and construct two kinds of nonconforming cuboidal elements first, then show that they satisfy some general conditions required to be locking-free. In section 4, three finite element schemes are presented and showed to be locking-free; optimal error estimates of them are obtained, uniformly for $\lambda \in (0, \infty)$. We end this paper with some numerical examples in the last section.

2. Preliminary

For isotropic and homogeneous materials, we consider the pure displacement boundary value problem of three-dimensional linear elasticity. Let $\Omega \in R^3$ be a bounded convex polyhedron with the boundary $\partial\Omega$. The displacement $\vec{u}(x) = (u_1(x), u_2(x), u_3(x))^T$ satisfies the following partial differential equation:

$$\begin{cases} -div\, \sigma(\vec{u}) = \vec{f}, & \text{in} \quad \Omega, \\ \vec{u} = \vec{0}, & \text{on} \quad \partial\,\Omega, \end{cases} \tag{2.1}$$

where $\vec{f} \in L^2(\Omega)^3$, and

$$\sigma(\vec{u}) = \mu \left(\nabla\vec{u} + (\nabla\vec{u})^T\right) + \lambda\, div\vec{u}\, I, \qquad div\vec{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z},$$

and I is the identity. (2.1) is equivalent to the following boundary value problem

$$\begin{cases} -\mu\Delta\vec{u} - (\mu + \lambda)\nabla(div\vec{u}) = \vec{f} & \text{in} \quad \Omega, \\ \vec{u} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.2}$$

whose equivalent weak form is

$$\begin{cases} \text{Find} \quad \vec{u} \in V, & \text{such that} \\ a(\vec{u}, \vec{v}) = (\vec{f}, \vec{v}) & \forall \vec{v} \in V, \end{cases} \tag{2.3}$$

where $V = H_0^1(\Omega)^3$,

$$a(\vec{u}, \vec{v}) \;=\; \int_\Omega \Big\{ \mu \sum_{i=1}^{3} \nabla u_i \cdot \nabla v_i + (\mu + \lambda)(div\vec{u})(div\vec{v}) \Big\} dx, \tag{2.4}$$

$$(\vec{f}, \vec{v}) \;=\; \int_\Omega \vec{f} \cdot \vec{v} dx, \tag{2.5}$$

and $\lambda \in (0, \infty)$, $\mu \in [\mu_1, \mu_2]$, $0 < \mu_1 < \mu_2$ are Lamé constants.

It is easy to see that the bilinear form in (2.4) is symmetric, coercive and continuous on V by Poincáre's inequality and the Lax-Milgram Theorem. So there exists a unique solution of (2.3). To analyze the convergence of our finite element schemes, we need the following assumption:

Proposition 2.1. *Assume $\Omega \subset R^3$ is a convex polyhedron. $\vec{u}$ is the solution of (2.1) or (2.2). Then the following regularity of $\vec{u}$ is true:*

$$\|\vec{u}\|_{2,\Omega} + \lambda \|div\vec{u}\|_{1,\Omega} \le C\|\vec{f}\|_{0,\Omega}, \tag{2.6}$$

where C is a positive constant independent of λ.

Remark 2.2. Proposition 2.1 is true for the planar elasticity(see [2][3]). But as for the three-dimensional case, we have not found the result to the best our knowledge. Since the proof of (2.6) is far more difficult than that of the planar case and beyond the object of this paper, we use it as an assumption and do not attempt to prove it. A rougher result is (see Theorem 6.3-6