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THE UPWIND FINITE ELEMENT SCHEME AND MAXIMUM PRINCIPLE FOR NONLINEAR CONVECTION-DIFFUSION PROBLEM *1)

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Abstract

In this paper, a kind of partial upwind finite element scheme is studied for two-dimensional nonlinear convection-diffusion problem. Nonlinear convection term approximated by partial upwind finite element method considered over a mesh dual to the triangular grid, whereas the nonlinear diffusion term approximated by Galerkin method. A linearized partial upwind finite element scheme and a higher order accuracy scheme are constructed respectively. It is shown that the numerical solutions of these schemes preserve discrete maximum principle. The convergence and error estimate are also given for both schemes under some assumptions. The numerical results show that these partial upwind finite element scheme are feasible and accurate.

Mathematics subject classification: 65F10, 65N30.
Key words: Convection-diffusion problem, Partial upwind finite element, Maximum principle.

1. Introduction

Convection-diffusion processes appear in many areas of science and technology. For example, fluid dynamics, heat and mass transfer, hydrology and so on. This is the reason that the numerical solution of convection-diffusion problem attracts a number of speciality. From an extensive literature devoted to linear problems, let us mentioned some papers [2], [3], monographs[1], and the references therein, few approaches to the solution of nonlinear problems mentioned in the papers [4], [10] and [11].

It is a well-known fact that the use of a classical Galerkin method with continuous piecewise linear finite elements leads to spurious oscillations when the local Péclet number is large. To obtain an effective scheme in the case of that convection term is dominate or the Peclét number is large, it is required to consider a suitable approximate for the convection term $\nabla \cdot \vec{b}(u)$. The partial upwind finite element scheme is known as the method solve convection-diffusion problem when the convection term is dominated [3]. In [10], the partial upwind finite element scheme for two-dimensional nonlinear Burgers equation is studied. In [4] and [11], M. Feistauer and his fellows investigates a combined finite volume-finite element methods for two-dimensional nonlinear convection-diffusion problem which the convection term only is nonlinear, and the convection term is explicit scheme. The purpose of this paper is to present an partial upwind finite element scheme for a more general type of two-dimensional nonlinear convection-diffusion problem which approximate the diffusion term by standard Galerkin method and approximate the convection term by partial upwind finite element method on the mesh dual to the triangular grid of weakly acute type.

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The method is easy to be carried out and it is applicable in multi-dimensional problem. Especially, it preserve maximum principle of original problem. Under some assumptions on the regularity of the exact solution of the continuous problem, we prove error estimates of the scheme. The numerical computations for the system of compressible viscous flow have demonstrated that the partial upwind finite element scheme is feasible and produces numerical results which are very promising.

This paper consists of seven sections. In Section 2, the notation and the nonlinear problem is given. In Section 3, the finite element space is defined, and the partial upwind finite element scheme. The discrete maximum principle and the convergence of the scheme is shown in Section 4 and Section 5 respectively. On the base of above work, a higher order accuracy scheme is studied in Section 6. In Section 7, we give another partial upwind finite element scheme, and prove its discrete maximum principle and convergence. To test above schemes, we give some numerical examples in Section 8, these numerical results show that these partial upwind finite element schemes are feasible and accurate.

2. Formulation of the Problem and Some Notations

Throughout this paper, we will use C (with or without subscript or superscript) to denote generic constant independent of discrete parameter. $W^{m,p}(\Omega)$ denotes usual Sobolev spaces, where $\Omega \subset R^2$ is a convex polygon domain, m, p are nonnegative integer. The corresponding norm and semi-norm are $\|\cdot\|_{m,p,\Omega}$ and $|\cdot|_{m,p,\Omega}$ [6]. Particular, for $p=2$, $H^m(\Omega) = W^{m,2}(\Omega)$, the corresponding norm and semi-norm are $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$ respectively. Let $(\cdot,\cdot)$ denotes the inner product of $L_2(\Omega)$, then

$$(u,v) = \int_\Omega uv\mathrm{d}x, \quad \|u\|_{0,\Omega} = (u,u).$$

As usual $H_0^1(\Omega) = \{v \in H^1(\Omega); v|_{\partial\Omega} = 0\}$ denotes the subspace of $H^1(\Omega)$.

We consider the following two-dimensional nonlinear convection-diffusion initial boundary problem (P)

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nabla\cdot(a(u)\nabla u) + \nabla\cdot\vec{b}(u) = f(u) & (x,t) \in \Omega\times[0,T] = D; & (2.1)\\ u(x,t) = 0 & (x,t) \in \Gamma\times[0,T]; & (2.2)\\ u(x,0) = u^0(x) & x \in \bar{\Omega}. & (2.3)\end{cases}$$

where Γ is the boundary of Ω, $x = (x_1, x_2)$.

We define the bound set on $\mathbf{R}$:

$$G = \{u : |u| \le K_0\}$$

where K_0 is a positive constant which will be fixed later.

We assume the coefficient of problem (P) satisfied the following condition:

(**A1**) There exist constants m, M_a, C_1 and C_2 which depend on K_0 such that

$$0 < m < a(u) < M_a, \quad \forall (x,t) \in \Omega\times(0,T], u \in G.$$

$$|f(u)| \le C_1|u| + C_2, \quad \forall (x,t) \in \Omega\times(0,T], u \in \mathbf{R}.$$

$\vec{b}(u) = \big(b^{(1)}(u), b^{(2)}(u)\big) \in W_\infty^1(G)\times W_\infty^1(G)$, $f(u) \in W_\infty^1(G\times\Omega\times(0,T])$, $u^0(x) \in C(\bar{\Omega}) \bigcap H_0^1(\Omega)$.

(**A2**) $a(u)$, $\vec{b}(u)$ and $f(u)$ are locally Lipschitz continuous

$$|a(u) - a(v)| \le L|u-v|, \quad \forall u, v \in G.$$