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COMPUTING A NEAREST P-SYMMETRIC NONNEGATIVE DEFINITE MATRIX UNDER LINEAR RESTRICTION *1)

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Abstract

Let P be an $n \times n$ symmetric orthogonal matrix. A real $n \times n$ matrix A is called P-symmetric nonnegative definite if A is symmetric nonnegative definite and $(PA)^T = PA$. This paper is concerned with a kind of inverse problem for P-symmetric nonnegative definite matrices: Given a real $n \times n$ matrix $\widetilde{A}$, real $n \times m$ matrices X and B, find an $n \times n$ P-symmetric nonnegative definite matrix A minimizing $\|A - \widetilde{A}\|_F$ subject to $AX = B$. Necessary and sufficient conditions are presented for the solvability of the problem. The expression of the solution to the problem is given. These results are applied to solve an inverse eigenvalue problem for P-symmetric nonnegative definite matrices.

Mathematics subject classification: 15A24, 65F20.
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1. Introduction

Throughout this paper, we denote the real $m \times n$ matrix space by $\mathbf{R}^{m\times n}$, the set of all orthogonal matrices in $\mathbf{R}^{n\times n}$ by $\mathbf{OR}^{n\times n}$, the transpose of a real matrix A by A^T, the Moore-Penrose pseudoinverse of a matrix A by A^+, the $n \times n$ identity matrix by I_n, the set of all symmetric nonnegative definite matrices in $\mathbf{R}^{n\times n}$ by $\mathbf{SR}_0^{n\times n}$. $A > 0 (A \geq 0)$ means that A is a real symmetric positive(nonnegative) definite matrix. For $A, B \in \mathbf{R}^{m\times n}$, we define an inner product in $\mathbf{R}^{m\times n} :< A, B >= tr(B^T A)$, then $\mathbf{R}^{m\times n}$ is a Hilbert space. The matrix norm $\|\cdot\|_F$ induced by the inner product is the Frobenius norm.

Definition 1.1 (c.f.[5]). *A real $n \times n$ matrix $A = (a_{ij}) \in \mathbf{R}^{n\times n}$ is called doubly symmetric or bisymmetric if*

$$a_{ij} = a_{ji} = a_{n+1-j,n+1-i}, \quad i = 1, 2, \cdots, n.$$

The set of all bisymmetric matrices in $\mathbf{R}^{n\times n}$ is denoted by $\mathbf{BSR}^{n\times n}$. A real $n \times n$ matrix A is said to be bisymmetric nonnegative definite if A is bisymmetric and nonnegative definite. The set of all bisymmetric nonnegative definite matrices in $\mathbf{R}^{n\times n}$ is denoted by $\mathbf{BSR}_0^{n\times n}$.

Definition 1.2. *Let $P \in \mathbf{R}^{n\times n}$ be a symmetric orthogonal matrix. $A \in \mathbf{R}^{n\times n}$ is called P-symmetric nonnegative definite matrix if A is symmetric nonnegative definite and $(PA)^T = PA$. The set of all P-symmetric nonnegative definite matrices in $\mathbf{R}^{n\times n}$ is denoted by $\mathbf{SR}_P^{n\times n}$.*

If $P = I_n$, then $\mathbf{SR}_{I_n}^{n\times n} = \mathbf{SR}_0^{n\times n}$. e_i denotes the ith column of I_n. Let $S_n = [e_n, e_{n-1}, \cdots, e_1]$. If $P = S_n$, then $\mathbf{SR}_{S_n}^{n\times n} = \mathbf{BSR}_0^{n\times n}$.

In this paper, we consider the following problem.

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Problem IP. *Given a matrix $\widetilde{A} \in \mathbf{R}^{n\times n}$, two matrices $X, B \in \mathbf{R}^{n\times m}$, let*

$$S_A = \{A \in \mathbf{SR}_P^{n\times n} | AX = B\}, \tag{1.1}$$

find $\widehat{A} \in S_A$ such that

$$\|\widetilde{A} - \widehat{A}\|_F = \inf_{\forall A \in S_A} \|\widetilde{A} - A\|_F. \tag{1.2}$$

Problem IP is essentially computing the nearest P-symmetric nonnegative definite matrix in the Frobenius norm to an arbitrary real matrix $\widetilde{A}$ under the linear restriction $AX = B$. The problem arises in a remarkable variety of applications such as structural modification and system identification[4,11]. If $P = I_n$, Problem IP reduces to an inverse problem for real symmetric nonnegative definite matrices[12]. If $P = S_n$, Problem IP is an inverse problem for bisymmetric nonnegative definite matrices[10]. If $B = X\Lambda, \Lambda = diag(\lambda_1, \cdots, \lambda_m) \in \mathbf{R}^{m\times m}$, then the set S_A, further the solution $\widehat{A}$, is determined by the eigenvalues $\lambda_1, \cdots, \lambda_m$ and corresponding eigenvectors, Problem IP becomes an inverse eigenvalue problem[2].

The symmetric nonnegative definite solutions to the matrix inverse problem $AX = B$ were studied by Zhang[12]. The analysis of Re-positive and Re-nonnegative definite solutions can be found in [8] and [9], respectively. The bisymmetric nonnegative definite solutions to the matrix inverse problem $AX = B$ were treated in [10]. In this paper, the results from [10] are generalized and extended to P-symmetric nonnegative definite matrices.

In section 2, we give necessary and sufficient conditions for the set S_A to be nonempty and construct the set S_A explicitly when it is nonempty. In section 3, we show that there exists a unique solution in Problem IP if the set S_A is nonempty and present the expression of the solution to Problem IP. In section 4, we consider an inverse eigenvalue problem for P-symmetric nonnegative definite matrices.

2. The Set $\mathbf{S_A}$

To begin with, we introduce a lemma.

Lemma 2.1 (c.f.[7]). *Suppose that $P \in \mathbf{OR}^{n\times n}$ is symmetric. Then there exists an orthogonal matrix $U \in \mathbf{OR}^{n\times n}$ such that*

$$P = U \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix} U^T. \tag{2.1}$$

The representation (2.1) is referred to as a spectral decomposition of the matrix P. For convenience, let us introduce the notations

$$k_1 = k, \quad k_2 = n - k.$$

It is easy to obtain the following lemma from Definition 1.2.

Lemma 2.2. *$A \in \mathbf{SR}_P^{n\times n}$ if and only if*

$$A^T = A \geq 0, \quad AP - PA = 0. \tag{2.2}$$

About the structure of $\mathbf{SR}_P^{n\times n}$, we have the following result.

Theorem 2.1. *Let the spectral decomposition of the matrix $P \in \mathbf{OR}^{n\times n}$ be (2.1), $A \in \mathbf{SR}_P^{n\times n}$ if and only if*

$$A = U \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} U^T, \tag{2.3}$$

where $A_{ii} \in \mathbf{SR}_0^{k_i\times k_i} (i = 1, 2)$.