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LONG-TIME BEHAVIOR OF FINITE DIFFERENCE SOLUTIONS OF THREE-DIMENSIONAL NONLINEAR SCHRÖDINGER EQUATION WITH WEAKLY DAMPED [*1)]

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Abstract

The three-dimensional nonlinear Schrödinger equation with weakly damped that possesses a global attractor are considered. The dynamical properties of the discrete dynamical system which generate by a class of finite difference scheme are analysed. The existence of global attractor is proved for the discrete dynamical system.

Mathematics subject classification: 65M06, 35Q55, 65P99.
Key words: Nonlinear Schrödinger equation, Finite difference method, Global attractor.

1. Introduction

The three-dimensional nonlinear schrödinger equation with weakly damped

$$i\frac{\partial u}{\partial t} + \triangle u + g(|u|^2)u + i\gamma u = f \quad x \in \Omega,\ t > 0 \tag{1.1}$$

where $i = \sqrt{-1}, \triangle u = \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2}, \gamma > 0, \Omega = (0, L_1)\times(0, L_2)\times(0, L_3)$, together with appropriate boundary and initial conditions, is arisen in many physical fields. The existence of an attractor is one of the most important characteristics for a dissipative system. The long-time dynamics is completely determined by the attractor of the system. J.M. Ghidaglia [2] studied the long-time behavior of the nonlinear Schrödinger equation (1.1) in dimension one and proved the existence of a compact global attractor $\mathcal{A}$ in H^1 which has the finite Hausdorff and fractal dimension under some conditions. The equation (1.1) in dimension three were studied by P. Laurencot[6], S. Y. Wu & Y. Zhao[9], and obtain also the existence of a compact global attractor $\mathcal{A}$ in H^1 under conditions (1.4)-(1.6). Guo Boling[3] construct the approximate inertial manifolds for the equation (1.1) and the order of approximation of these manifolds to the global attractor were derived. At the same time, a semidiscrete finite difference method of the equation was discussed by Yin Yan[10] and we studied also long-time behavior of completely discrete finite difference solutions of the equation in dimension one in [11]. In this paper, a completely discrete scheme is discussed by finite difference method for the equation (1.1) in dimension three with initial condition

$$u(x, 0) = u_0(x),\ x \in \Omega \tag{1.2}$$

and Dirichlet boundary condition:

$$u|_{\partial\Omega} = 0, t \in R^+, \tag{1.3}$$

where $f \in C(\overline{\Omega})$, $g(s)(0 \le s < \infty)$ is a real valued smooth function that satisfies

$$\lim_{s\to+\infty} \frac{G_+(s)}{s^{\frac{5}{3}}} = 0, \tag{1.4}$$

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and for some $\omega > 0$

$$\lim_{s\to+\infty} sup\frac{h(s)-\omega G(s)}{s^{\frac{5}{3}}} \leq 0. \tag{1.5}$$

$$|g'(s)| \leq M, \quad s \in R^+. \tag{1.6}$$

where $h(s) = sg(s), G(s) = \int_0^s g(\sigma)d\sigma$ and $G_+(s) = max\{G(s), 0\}$.

In this paper we make the following assumptions on $g(s)$ besides the above conditions (1.4)–(1.6)

$$\lim_{s\to+\infty} \frac{G(s)}{s^{\frac{5}{3}}} = 0, \tag{1.4$'$}$$

$$g(s) \text{ and } g'(s) \text{ do not change sign in } R^+. \tag{1.7}$$

Finally, let us denote the first difference quotient of $G(s)$ by $G[s_2, s_1]$ on points s_1, s_2, i.e.

$$G[s_2, s_1] = \begin{cases} \frac{G(s_2)-G(s_1)}{s_2-s_1}, & if\ s_2 \neq s_1, \\ g(s_1), & if\ s_2 = s_1. \end{cases}$$

The paper is organized as follows. In §2 we prove embedding theorems and interpolation inequalities for discrete(grid) functions, which are the analogues of the embedding theorems and interpolation inequalities for the Sobolev space $W^{m,p}(\Omega)$, and a discrete system is established by finite difference method for the continuous system which is generated by the nonlinear Schrödinger equations (1.1) with Dirichlet boundary condition (1.3) and initial condition (1.2). In §3 we study the existence of absorbing sets and attractor for the discrete system.

2. Finite Difference Scheme

First, let us divide the domain $\overline{\Omega}$ into small grids by the parallel planes $x_1 = ih_1(0 \leq i \leq J_1), x_2 = jh_2(0 \leq j \leq J_2)$ and $x_3 = kh_3(0 \leq k \leq J_3)$, where h_1, h_2, h_3 are the spatial mesh lengths, J_1, J_2, J_3 are positive integers, and $J_1h_1 = L_1, J_2h_2 = L_2, J_3h_3 = L_3$. Denote the real or the complex value discrete functions on the grid point set $\overline{\Omega}_h = \{(ih_1, jh_2, kh_3); 0 \leq i \leq J_1, 0 \leq j \leq J_2, 0 \leq k \leq J_3\}$ by $\phi, \psi, \cdots$, and let $\Omega_h = \overline{\Omega}_h \cap \Omega, \partial\Omega_h = \overline{\Omega}_h \cap \partial\Omega$. We employ $\triangle_{+h_l}, \triangle_{-h_l}$ and δ_{h_l} to denote the forward difference, the backward difference and the forward difference quotient operators respectively in $x_l(1 \leq l \leq 3)$ direction, and $\triangle_h$ to denote the discrete Laplace operator, i.e.

$$\triangle_h \phi_{i,j,k} = \sum_{l=1}^{3} \frac{\triangle_{+h_l}\triangle_{-h_l}\phi_{i,j,k}}{h_l^2} \ .$$

We let $\triangle t$ denote the temporal mesh length, $J = (J_1+1)(J_2+1)(J_3+1)$, $h = \max_{1\leq l\leq 3} h_l$, and assume that there exists a positive constant $\delta \in (0, 1]$, such that $\delta h \leq h_l(l = 1, 2, 3)$.

We introduce the discrete L^2 inner product

$$(\phi, \psi)_h = \sum_{i=0}^{J_1}\sum_{j=0}^{J_2}\sum_{k=0}^{J_3} \phi_{i,j,k}\overline{\psi}_{i,j,k}h_1h_2h_3$$

and the discrete H^1 inner product

$$(\phi, \psi)_{1,h} = \sum_{i=0}^{J_1-1}\sum_{j=0}^{J_2-1}\sum_{k=0}^{J_3-1}\sum_{l=1}^{3} \delta_{h_l}\phi_{i,j,k}\overline{\delta_{h_l}\psi}_{i,j,k}h_1h_2h_3,$$

together with the associated norms

$$\|\phi\|_h = (\phi, \phi)_h^{\frac{1}{2}}, \quad \|\phi\|_{1,h} = (\phi, \phi)_{1,h}^{\frac{1}{2}}.$$