Journal of Computational Mathematics, Vol.22, No.4, 2004, 567–580.

ON THE GENERAL ALGEBRAIC INVERSE EIGENVALUE PROBLEMS *

Yu-hai Zhang
(*Department of Mathematics, Shandong University, Jinan 250100, China*)
(*ICMSEC, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, China*)

Abstract

A number of new results on sufficient conditions for the solvability and numerical algorithms of the following general algebraic inverse eigenvalue problem are obtained: Given $n+1$ real $n\times n$ matrices $A=(a_{ij}),\ A_k=(a_{ij}^{(k)})(k=1,2,\dots,n)$ and n distinct real numbers $\lambda_1,\lambda_2,\dots,\lambda_n$, find n real numbers $c_1,c_2,\dots,c_n$ such that the matrix $A(c)=A+\sum\limits_{k=1}^{n}c_kA_k$ has eigenvalues $\lambda_1,\lambda_2,\dots,\lambda_n$.

Mathematics subject classification: 15A18, 34A55.
Key words: Linear algebra, Eigenvalue problem, Inverse problem.

1. Introduction

We are interested in solving the following inverse eigenvalue problems:

Problem A(Additive inverse eigenvalue problem). Given an $n\times n$ real matrix $A=(a_{ij})$, and n distinct real numbers $\lambda_1,\lambda_2,\dots,\lambda_n$, find a real $n\times n$ diagonal matrix $D=diag(c_1,c_2,\dots,c_n)$ such that the matrix $A+D$ has eigenvalues $\lambda_1,\lambda_2,\dots,\lambda_n$.

Problem M(Multiplicative inverse eigenvalue problem). Given an $n\times n$ real matrix $A=(a_{ij})$, and n distinct real numbers $\lambda_1,\lambda_2,\dots,\lambda_n$, find a real $n\times n$ diagonal matrix $D=diag(c_1,c_2,\dots,c_n)$ such that the matrix DA has eigenvalues $\lambda_1,\lambda_2,\dots,\lambda_n$.

Problem G(General inverse eigenvalue problem). Given $n+1$ real $n\times n$ matrices $A=(a_{ij}),\ A_k=(a_{ij}^{(k)})(k=1,2,\dots,n)$ and n distinct real numbers $\lambda_1,\lambda_2,\dots,\lambda_n$, find n real numbers $c_1,c_2,\dots,c_n$ such that the matrix $A(c)=A+\sum\limits_{k=1}^{n}c_kA_k$ has eigenvalues $\lambda_1,\lambda_2,\dots,\lambda_n$.

Evidently Problem **A** and **M** are special cases of Problem **G**. The solutions of Problem **G** are complicated. A number of results on sufficient conditions for the solvability, stability analysis of solution and numerical algorithms of Problem **G** with real symmetric matrices can be found in [1,3,11,12,14,16,19,20,21,22]. These results are all obtained by studying the following nonlinear system

$$\lambda_i(A(c))=\lambda_i,\ \ i=1,2,\dots,n \tag{1}$$

where $\lambda_i(A(c))$ is the ith eigenvalue of $A(c)$, or

$$det(A(c)-\lambda_iI)=0,\quad i=1,2,\dots,n. \tag{2}$$

Most numerical algorithms depend heavily on the fact that the eigenvalues of real symmetric matrix are real valued and, hence, can be totally ordered[13]. But non-symmetric matrices have not the fact. Less results on non-symmetric problems can be found. In this paper, we

* Received April 17, 2002.

use another approach to investigate Problem **G**. The main idea is to treat Problem **G** as the following equivalent problem.

$$A(c)T = T\Lambda \tag{3}$$

where $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and T is a non-singular matrix. We see that the columns of T are the eigenvectors of $A(c)$. (3) is equivalent to a polynomial system(see Section 2). It is not necessary to consider ordering eigenvalues to solve the polynomial system.

In Section 2 it is proved that problem **G** is equivalent to a polynomial system. In Section 3 by studying the system with the help of Brouwer's fixed point theorem we obtain some new sufficient conditions on the solvability , which improve the results in[1,3,5,8,9]. In Section 4, we propose a linearly convergent iterative algorithm and a quadratically convergent iterative algorithm. Several examples are given in this paper.

Throughout this paper we use the following notation. Let $R^{n\times n}$ be the set of all $n \times n$ real matrices. $R^n = R^{n\times 1}$. Let

$$h_i^{(k)} = \sum_{j=1,\neq i}^{n} |a_{ij}^{(k)}|, \quad h_i = \sum_{k=1}^{n} h_i^{(k)}, \quad H = (h_i^{(k)}) \in R^{n\times n}.$$

Obviously, H is a nonnegative matrix. Let $\rho(H)$ be the spectral radius of H.

For a permutation π of the n items $\{1, \ldots, n\}$, let

$$s_{ij} = a_{ij} + \sum_{k=1}^{n} (\lambda_{\pi(k)} - a_{\pi(k),\pi(k)}) a_{ij}^{(k)}, \quad l_{ij} = |s_{ij}|, \quad i, j = 1, 2, \ldots, n, \ i \neq j \tag{4}$$

$$l_i = \sum_{j=1,\neq i}^{n} l_{ij}, \quad i = 1, 2, \ldots, n \tag{5}$$

2. Equivalent Polynomial System

Without loss of generality we can suppose that[1,3,8,9] $a_{ii} = 0 (i = 1, \ldots, n)$ in Problem **A**, $a_{ii} = 1 (i = 1, \ldots, n)$ in Problem **M**, and $a_{ii}^{(k)} = \delta_{ik} (i, k = 1, \ldots, n)$ in Problem **G**.

Theorem 1. *Problem* **G** *has a solution $c_1, c_2, \ldots, c_n \in R$ if and only if there exists a permutation π of the n items $\{1, \ldots, n\}$ such that the following polynomial system*

$$\begin{cases} (\lambda_{\pi(j)} - a_{ii} - c_i) t_{ij} = (a_{ij} + \sum\limits_{k=1}^{n} c_k a_{ij}^{(k)}) + \sum\limits_{l=1,\neq i,j}^{n} (a_{il} + \sum\limits_{k=1}^{n} c_k a_{il}^{(k)}) t_{lj}, \quad i, j = 1, \ldots, n, i \neq j \\ \lambda_{\pi(i)} - a_{ii} - c_i = \sum\limits_{l=1,\neq i}^{n} (a_{il} + \sum\limits_{k=1}^{n} c_k a_{il}^{(k)}) t_{li}, \quad i = 1, \ldots, n \end{cases} \tag{6}$$

has a solution $c_i \in R$, $t_{ij} \in R$ $(i, j = 1, \ldots, n, \ i \neq j)$.

Proof. Suppose Problem **G** has a solution $c = (c_1, c_2, \ldots, c_n)^T \in R^n$. Since the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $A(c)$ are all different , the Jordan canonical form of $A(c)$ is $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$, and therefore there exists a nonsingular matrix $S = (s_{ij}) \in C^{n\times n}$ such that

$$A(c) = S\Lambda S^{-1},$$

that is

$$A(c)S = S\Lambda. \tag{7}$$

Noting that $A(c)$ is a real matrix only with real eigenvalues, then the similarity matrix S can be taken to be real. Notice that $S \in R^{n\times n}$ is nonsingular, hence $detS \neq 0$, then there exists a