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THE FINITE ELEMENT ANALYSIS OF THE CONTROLLED-SOURCE ELECTROMAGNETIC INDUCTION PROBLEMS BY FRACTIONAL-STEP PROJECTION METHOD *1)

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Abstract

This paper provides an convergence analysis of a fractional-step projection method for the controlled-source electromagnetic induction problems in heterogenous electrically conducting media by means of finite element approximations. Error estimates in finite time are given. And it is verified that provided the time step τ is sufficiently small, the proposed algorithm yields for finite time T an error of $\mathcal{O}(h^s+\tau)$ in the L^2-norm for the magnetic field $\boldsymbol{H}$, where h is the mesh size and $1/2<s\leq 1$.

Mathematics subject classification: 65N30.
Key words: Controlled-source electromagnetic induction problems, Fractional-step projection method, Finite element, Error analysis.

1. Introduction

The numerical treatment of the controlled-source electromagnetic(CSEM) induction problems, which are widely applied in geophysical prospecting, have received much attention in the last decades (see [5, 11, 16, 17, ?]). So-called CSEM problems are actually that electric and magnetic fields at low frequencies (such that displacement currents can be neglected) satisfy the diffusive Maxwell's equations:

$$\nabla\times\boldsymbol{E}+\mu_0\frac{\partial\boldsymbol{H}}{\partial t}=\boldsymbol{0}, \tag{1.1}$$

$$\nabla\times\boldsymbol{H}-\sigma\boldsymbol{E}=\boldsymbol{J}_s, \tag{1.2}$$

where μ_0 is the magnetic permeability of free space, σ is the spatially varying electrical conductivity of the geological formation being studied with $0<\bar{\sigma}\leq\sigma(\boldsymbol{x})$, and $\boldsymbol{J}_s(\boldsymbol{x},t)$ is source electric current density.

By equations (1.1) and (1.2), eliminating electric field $\boldsymbol{E}$ we obtain

$$\mu_0\frac{\partial\boldsymbol{H}}{\partial t}+\nabla\times(\sigma^{-1}\nabla\times\boldsymbol{H})=\nabla\times(\sigma^{-1}\boldsymbol{J}_s). \tag{1.3}$$

A constitutive equation $\boldsymbol{B}=\mu_0\boldsymbol{H}$ relates the magnetic induction and magnetic field vectors. The divergence-free condition

$$\nabla\cdot\boldsymbol{B}=0\quad\Longrightarrow\quad\nabla\cdot\boldsymbol{H}=0 \tag{1.4}$$

is also imposed, indicating that no magnetic induction exists inside the solution domain Ω.

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For convenience of numerical treatment, we add a multiplier term $\nabla\phi$ to the left side of equation (1.3) to obtain

$$\mu_0 \frac{\partial \boldsymbol{H}}{\partial t} + \nabla \times (\sigma^{-1} \nabla \times \boldsymbol{H}) + \nabla\phi = \nabla \times (\sigma^{-1} \boldsymbol{J}_s). \tag{1.5}$$

By equation (1.4), we take the divergence in two side of above equation to get

$$\nabla^2 \phi = 0. \tag{1.6}$$

It can easily be founded that if we enforce a zero Dirichlet condition on ϕ along the boundary Γ of the solution domain Ω, then the multiplier function ϕ added in equation (1.5) is identically zero value on that domain. Thus, equation (1.5) is equivalent to equation (1.3).

In following context we shall concentrate our attention on the finite element approximation of the following initial-boundary value problem of equations (1.4) and (1.5):

$$\begin{cases} \mu_0 \dfrac{\partial \boldsymbol{H}}{\partial t} + \nabla \times (\sigma^{-1} \nabla \times \boldsymbol{H}) + \nabla\phi = \boldsymbol{F}, \quad \Omega \times (0,T), \\ \nabla \cdot \boldsymbol{H} = 0, \quad \Omega \times [0,T], \\ \boldsymbol{H} \times \boldsymbol{n} = \boldsymbol{0}, \quad \partial\Omega \times [0,T], \\ \boldsymbol{H}(\cdot, 0) = \boldsymbol{H}_0, \quad \Omega. \end{cases} \tag{1.7}$$

Here vector value function $\boldsymbol{F} = \nabla \times (\sigma^{-1} \boldsymbol{J}_s)$ and Ω is a bounded, simply-connected polyhedral domain with connected boundary $\Gamma = \partial\Omega$ and $\boldsymbol{n}$ the unit normal vector to Γ.

As is known to all, the fractional-step projection method of Chorin [6, 7] and Shen [22, 23] and Guermond [1, 12] has been successfully applied to solve the incompressible Navier-Stokes equations in primitive variables in recent years. This method is based on rather peculiar time-discretization of the Navier-Stokes equations, in which the convection-diffusion and the incompressibility are dealt with in two different substeps and therefore the original problem is converted into solving a convection-diffusion problem and a Poisson problem at each step. Thus, the projection method has advantage of much lower amount of computation in comparison with coupled techniques such as those that are based on the Uzawa operator (see [2, 3, 14] et al).

In view of above-mentioned virtues, the aim of this paper is to present and analyze a fractional-step projection algorithm for the controlled-source electromagnetic induction problem in heterogenous electrically conducting media by means of finite element approximations.

The remainder of this paper is organized as follows. Some preliminary results is stated and the fractional-step projection scheme is proposed in Section 2. Section 3 devotes to the error estimates with mild regularity assumptions on the solution of the continuous problems.

2. Fractional-step Projection Scheme

Firstly, we state some preliminary knowledge which will be frequently cited in the sequel. Throughout this paper we assume that $\Omega \subset \Re^3$ is a sufficiently smooth bounded, simply connected polyhedral domain with connected boundary $\Gamma = \partial\Omega$ and $\boldsymbol{n}$ is the unit normal vector to Γ. As usual, $W^{s,\,p}(\Omega)$ denotes the real Sobolev space, $0 \le s < \infty$, $0 \le p \le \infty$, equipped with the norm $\|\cdot\|_{s,\,p}$ and semi-norm $|\cdot|_{s,\,p}$. The space $W_0^{s,\,p}$ is the completion of the space of smooth functions compactly supported in Ω with respect to the $\|\cdot\|_{s,\,p}$ norm (see [8, 13, 20]). For $p = 2$,we denote the Hilbert spaces $W^{s,\,2}(\Omega)$ (resp., $W_0^{s,\,2}(\Omega)$) by $H^s(\Omega)$ (resp., $H_0^s(\Omega)$). The related norm is denoted by $\|\cdot\|_s$. The dual space of $H_0^s(\Omega)$ is denoted by $H^{-s}(\Omega)$. For a fixed positive real number T, and a Banach space X, we denote by $L^p(X)$, $H^s(X)$ and $C(X)$ the space $L^p(0,T;X)$, $H^s(0,T;X)$ and $C(0,T;X)$, respectively.