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ASYMPTOTIC STABILITY OF RUNGE-KUTTA METHODS FOR THE PANTOGRAPH EQUATIONS *1)

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Abstract

This paper considers the asymptotic stability analysis of both exact and numerical solutions of the following neutral delay differential equation with pantograph delay.

$$\begin{cases} x'(t) + Bx(t) + Cx'(qt) + Dx(qt) = 0, & t > 0, \\ x(0) = x_0, \end{cases}$$

where $B, C, D \in \mathcal{C}^{d\times d}, q \in (0, 1)$, and B is regular. After transforming the above equation to non-automatic neutral equation with constant delay, we determine sufficient conditions for the asymptotic stability of the zero solution. Furthermore, we focus on the asymptotic stability behavior of Runge-Kutta method with variable stepsize. It is proved that a L-stable Runge-Kutta method can preserve the above-mentioned stability properties.

Mathematics subject classification: 65L02, 65L05, 65L20.
Key words: Neutral delay differential equations, Pantograph delay, Asymptotic stability, Runge-Kutta methods, L-stable.

1. Introduction

Delay differential equations of neutral type provide a mathematical instrument to applied science[1]. Especially, it exerts important effect on investigating several electromagnetic problems. The general functional differential equation is given by

$$x'(t) = f\left(t, x(t), x'(\alpha(t)), x(\alpha(t))\right).$$

A classical case $\alpha(t) = t - \tau$ of such system has been recently considered by a lot of authors (for example, Kuang et al.[2] and Hu and Mitsui [3]). What's more, another interesting case which is far different from the previous, is the pantograph equation

$$\begin{cases} x'(t) = f(t, x(t), x'(qt), x(qt)), t > 0, \\ x(0) = x_0. \end{cases} \tag{1.1}$$

Where f is a given function, $0 < q < 1$, and $x(t)$ is unknown for $t > 0$. There are many applications for (1.1) both in electrodynamics and in the collection of current by the pantograph of an electric locomotive [4, 5]. We are interested in the investigation of the qualitative properties of equation (1.1). To this purpose we restrict ourselves to the special form of equation (1.1) given by

$$\begin{cases} x'(t) + Bx(t) + Cx'(qt) + Dx(qt) = 0, t > 0, \\ x(0) = x_0, \end{cases} \tag{1.2}$$

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where B, C, D are constant $d \times d$ complex matrices, $0 < q < 1$, and B is regular.

The asymptotic behaviour of the pantograph equation (1.2) has been analyzed, often in a simplified form,by a number of authors.Concerning pantograph equation

$$\begin{cases} x'(t) = ax(t) + bx(qt) + cx'(qt), t > 0, \\ x(0) = x_0, \end{cases} \tag{1.3}$$

where $a, b, c \in \mathcal{C}$ and $0 < q < 1$, an analytical study occurred in the works of Kato and McLeod [6], Carr and Dyson [7], Iserles and Terjéki [8] and Iserles [9], in which Iserles and Terjéki [8] applied a transformation to examine the behavior of the exact solutions, and Iserles [9] applied *Dirichlet series* to equation (1.3) for considering whether the exact solution displayed inside and on its stability boundary. In the present paper, we investigate the stability of exact solution of equation(1.2) by transforming equation(1.2) to a non-automatic neutral equation with constant delay and prove the contractivity and asymptotic stability by norm assessing. Related ideas of reformulating the problem for studying the asymptotic stability of the solutions have been considered in some recent papers such as in [8]. Actually Bellen et al.[10] has been used norm assessing to consider the analytical stability of equation

$$\begin{cases} x'(t) = Lx(t) + M(t)x(t - \tau(t)) + N(t)x'(t - \tau(t)), t > t_0, \\ x(t) = g(t), t \le t_0, \end{cases}$$

but we consider that the coefficient of $x(t)$ is matrix-value function like $M(t)$ and $N(t)$.

On the other hand, the investigations of numerical stability for (1.3) can be found in many papers, such as Buhmann, Iserles and Nørsett [11], Buhmann and Iserles [12, 13] and Liu [14], in which [12] considered a special case when q is a reciprocal of an integer and [14] gave an extension of this analysis to θ-methods by transforming the equation under consideration into a neutral equation with constant time lags. Moreover, Koto [15] dealt with stability of Runge-Kutta methods applied to the equation which is obtained from the equation (1.2) by the same change of the independent variable with [14]

$$y(t) = x(e^t).$$

In 1997, Bellen[16] applied the θ-methods with variable stepsize to (1.3). It is proved that θ-methods are asymptotically stable iff $\theta > 1/2$, which provided the subsequent research with a new idea. In the present paper, the Runge-Kutta methods with variable stepsize is applied to equation(1.2) and asymptotic stability of numerical solution is probed by investigating the Schur polynomial of perturbed equation.

This paper is organized as follows. In Section 2, the sufficient conditions both for the contractivity and for asymptotic stability of the exact solutions are given. In Section 3, we apply Runge-Kutta methods with variable stepsize to equation(1.2) and obtain numerical solutions. In Section 4, the conclusion of numerically asymptotic stability is drawn.

2. Sufficient Conditions of Contractivity and Asymptotic Stability

2.1. Transformation of Equation Form

To eliminate dependence on the derivative of the solution, we transform (1.2) to a neutral equation with constant delay by conversion

$$y(t) = x(e^t),$$