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RECONSTRUCTION OF SCATTERED FIELD FROM FAR-FIELD BY REGULARIZATION *1)

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Abstract

In this paper, we consider an inverse scattering problem for an obstacle $D \subset \mathcal{R}^2$ with Robin boundary condition. By applying the point source, we give a regularizing method to recover the scattered field from the far-field pattern. Numerical implementations are also presented.

Mathematics subject classification: 35R30, 35J05, 76Q05.
Key words: Inverse scattering, Point source, Regularization, Numerics.

1. Introduction

Let $D \subset \mathcal{R}^2$ be a simply connected domain with C^2 boundary ∂D. The scattering of time-harmonic acoustic plane wave for the obstacle D with impedance boundary condition is modeled by an exterior boundary value problem for the Helmholtz equation. That is, for a given incident plane wave $u^i(x) = e^{ikx\cdot d}$, $d \in \Omega = \{\xi \in \mathcal{R}^2 : |\xi| = 1\}$, the total wave field $u = u^i + u^s \in H^1_{loc}(\mathcal{R}^2 \setminus \overline{D})$ satisfies

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in } \mathcal{R}^2 \setminus \overline{D} \\ \frac{\partial u}{\partial \nu} + ik\sigma(x)u = 0, & \text{on } \partial D \\ \frac{\partial u^s}{\partial r} - iku^s = O\left(\frac{1}{\sqrt{r}}\right), & r = |x| \longrightarrow \infty \end{cases} \tag{1.1}$$

where ν is the unit normal vector of ∂D directed into the exterior of D. $u^s(x)$ is the scattered wave corresponding to the incident wave $u^i(x)$. Assume that $0 < \sigma(x) \in C(\partial D)$, then by the result in [4], we know that there exists a unique solution for the forward scattering problem (1.1).

For the incident field $u^i(x) = e^{ikx\cdot d}$, the far-field pattern $u^\infty(d, \theta)$ of scattered wave $u^s(x)$ can be defined by

$$u^s(x) = \frac{e^{ik|x|}}{\sqrt{|x|}} \left\{ u^\infty(d, \theta) + O\left(\frac{1}{|x|}\right) \right\}, \qquad |x| \longrightarrow \infty,$$

where $\theta \in \Omega$.

For a direct scattering problem, it aims to determine the scattered field $u^s(x)$ as well as its far-field pattern, provided the scatterer and the incident waves are known. The inverse

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scattering problems cover different ranges of scattering. For example, the determination of ∂D with unknown $\sigma(x)$ from the far-field pattern $u^\infty(d,\theta)$ for $\theta, d \in \Omega$ has been investigated in [1]. For some special cases of this kind of problem corresponding to $\sigma(x)=0$ and $\sigma(x)=\infty$ respectively, we refer to [2], [3], [4], [6], [7], [8].

From the standard scattering theory, we know that there exists a one-to-one correspondence between $u^\infty(d,\theta)$ and $u^s(x)$ for any given $d\in\Omega$. It is known as early as in 40's that the far-field pattern uniquely determines the analytic scattered field in the exterior of the scatterer(Rellich lemma). However, the map $u^s|_{\partial B} \to u^\infty$ is generally compact in any reasonable function space, where the cycle $B(0,R)$ contains the scatterer D within it. This fact indicates that the determination of scattered field from its far-field pattern is ill-posed. To the author's knowledge, although the inverse scattering problems of determining scatterer have been researched thoroughly, the reconstruction of scattered field from its far-field pattern receive little attention, especially in the treatment of ill-posedness and the numerical implementation. The importance of this problem is due to the fact that construction of the near field data of scattered wave from far-field pattern is a crucial step, when one considers the inverse scattering problem by the Dirichlet-to-Neumann map([1]). An important development in this issue can be found in [9], where the stability of recovering $u^s(x)$ from $u^\infty(d,\theta)$ is established by applying the point-source method.

This paper deals with the numerical schemes of determining $u^s(x)$ from the error data of u^∞. Due to the ill-posedness, some regularizing technique should be introduced. More precisely, for known error data $u^\infty_\delta(d,\theta)$ of far-filed pattern with the error level $\delta>0$, i.e.,

$$\|u^\infty_\delta(d,\cdot)-u^\infty(d,\cdot)\|_{L^2(\Omega)} \le \delta,$$

our task is to recover $u^s(x)$ approximately from $u^\infty_\delta(d,\theta)$. Motivated by the basic idea in [9] for treating the inverse scattering problem with the Dirichlet boundary, we establish a reciprocity principle for the scattering problem with the Robin boundary firstly, then we propose a regularizing scheme to reconstruct the near-field from $u^\infty(d,\theta)$. Numerical results illustrating the inversion scheme are also given. Our numerics show that the near field data is very sensitive to the far-field pattern when one applies the point source method to recover near field, even though the stability of this method has been proved theoretically in [9].

Our paper is organized as follows:

- Section 2: Relation between near-field and far-field
- Section 3: Regularization method
- Section 4: Numerical implementations

2. Relation between Near-Field and Far-Field

In this section, we state the relation between far-field and near-field. This relation is led from the potential theory.

Denote by $\Phi(x,y)$ the fundamental solution to 2-D Helmholtz equation, i.e.,

$$\Phi(x,y)=\frac{i}{4}H_0^{(1)}(k|x-y|)$$

with $H_0^{(1)}$ the Hankel function and define

$$(\mathbf{K}'\psi)(x)=2\int_{\partial D}\frac{\partial\Phi(x,y)}{\partial\nu(x)}\psi(y)ds(y), \quad x\in\partial D,$$