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CONVERGENCE OF THE EXPLICIT DIFFERENCE SCHEME AND THE BINOMIAL TREE METHOD FOR AMERICAN OPTIONS [*1)]

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Abstract

This paper is concerned with numerical methods for American option pricing. We employ numerical analysis and the notion of viscosity solution to show uniform convergence of the explicit difference scheme and the binomial tree method. We also prove the existence and convergence of the optimal exercise boundaries in the above approximations.

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1. Introduction

In the probability theory, the Black-Scholes model for American option pricing belongs to the optimal stopping problems. On the other hand, in the viewpoint of PDE, it is a parabolic variational inequality. Consequently, roughly speaking, there are two kinds of numerical methods for American option pricing based on the probabilistic approach and finite difference respectively.

The binomial tree method, as a discrete time model, is the most common approach for pricing options. Amin and Khanna (1994), using the probabilistic approach, first provided a convergence proof of the binomial tree method for American options [1]. In essence, the binomial tree method belongs to the probabilistic one. However, it can be proved that the binomial tree method is consistent with an explicit difference scheme. By virtue of the notion of viscosity solutions, Barles and Souganidis (1991) presented a framework to prove the convergence of difference schemes for fully nonlinear PDE problems [3]. Jaillet etc. (1990) studied the Brennan-Schwartz algorithm for pricing American put option based on the framework of variational inequalities [9]. Lamberton (1993) showed the convergence of the resulting optimal exercise boundary (critical price) [11]. He also proved the convergence result within the probabilistic approach.

This paper will concentrate on the explicit difference scheme and the binomial tree method for American options. The main purpose is to prove the convergence of the above approximations by using numerical analysis and the notion of viscosity solution, especially in the case of American call option for which the approximate sequence is not uniformly bounded in l^{∞}-norm.

The remainder of this paper is organized as follows: In section 2, we recall the Black-Scholes model, the explicit difference scheme and the binomial tree method for American options. In section 3 we will concentrate on the explicit difference scheme and show the existence of optimal exercise boundary computed in the approximation of the explicit difference scheme. Section 4 is

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devoted to the convergence proofs of the explicit difference scheme and the approximate optimal exercise boundary. We extend the results of the explicit difference scheme to the binomial tree method in section 5.

2. Black-Scholes Model and Numerical Methods for American Options

The Black-Scholes model for American options with continuous dividend yield is the following:

$$\begin{cases} \min\left(-\dfrac{\partial V}{\partial t} - \dfrac{\sigma^2}{2}S^2\dfrac{\partial^2 V}{\partial S^2} - (r-q)S\dfrac{\partial V}{\partial S} + rV, V - \psi\right) = 0 \\ \hfill in\ [0,T)\times(0,\infty) \\ V(T,S) = \psi(S) \quad in\ (0,\infty), \end{cases} \tag{2.1}$$

where $\psi(S) = (S-E)^+$ (call option) or $\psi(S) = (E-S)^+$ (put option), $r > 0, q$ and σ represent the interest rate, dividend yield and volatility [8].

Using the simple transformations $u(x,t) = V(S,t)$, $S = e^x$, (2.1) is transformed into the following constant-coefficient problem

$$\begin{cases} \min\left(-\dfrac{\partial u}{\partial t} - \dfrac{\sigma^2}{2}\dfrac{\partial^2 u}{\partial x^2} - (r-q-\dfrac{\sigma^2}{2})\dfrac{\partial u}{\partial x} + ru, u - \varphi\right) = 0 \\ \hfill in\ [0,T)\times(-\infty,\infty) \\ u(T,x) = \varphi(x) \quad in\ (-\infty,\infty), \end{cases} \tag{2.2}$$

where $\varphi(x) = (e^x - E)^+$ (call option) or $\varphi(x) = (E - e^x)^+$ (put option).

We now present the explicit difference scheme for (2.2). Given mesh size $\Delta x, \Delta t > 0$, $N\Delta t = T$, let $Q = \{(n\Delta t, j\Delta x) : 0 \le n \le N,\ j \in Z\}$ stand for the lattice. U_j^n represents the value of numerical approximation at $(n\Delta t, j\Delta x)$ and $\varphi_j = \varphi(j\Delta x)$. Taking the explicit difference for time and the conventional difference discretization for space, we have

$$\min\left(-\frac{U_j^{n+1} - U_j^n}{\Delta t} - \frac{\sigma^2}{2}\frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{\Delta x^2}\right.$$
$$\left. -(r-q-\frac{\sigma^2}{2})\frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\Delta x} + rU_j^n, U_j^n - \varphi_j\right) = 0$$

or

$$\begin{aligned} U_j^n \quad = \quad & \max\left(\frac{1}{1+r\Delta t}\left((1-\frac{\sigma^2\Delta t}{\Delta x^2})U_j^{n+1} + \frac{\sigma^2\Delta t}{\Delta x^2}(\frac{1}{2} + \frac{\Delta x}{2\sigma^2}(r-q-\frac{\sigma^2}{2}))U_{j+1}^{n+1}\right.\right. \\ & \left.\left. + \frac{\sigma^2\Delta t}{\Delta x^2}(\frac{1}{2} - \frac{\Delta x}{2\sigma^2}(r-q-\frac{\sigma^2}{2}))U_{j-1}^{n+1}\right), \varphi_j\right), \end{aligned}$$

which is denoted by

$$U_j^n \widehat{=} \max\left(\frac{1}{1+r\Delta t}\left((1-\alpha)U_j^{n+1} + \alpha(aU_{j+1}^{n+1} + cU_{j-1}^{n+1})\right), \varphi_j\right), \tag{2.3}$$

where

$$\alpha = \sigma^2\frac{\Delta t}{\Delta x^2},\ a = \frac{1}{2} + \frac{\Delta x}{2\sigma^2}(r-q-\frac{\sigma^2}{2}),\ c = 1-a.$$

By putting $\alpha = 1$ in (2.3), namely $\sigma^2\Delta t/\Delta x^2 = 1$, we get

$$U_j^n = \max\left(\frac{1}{1+r\Delta t}\left(aU_{j+1}^{n+1} + cU_{j-1}^{n+1}\right), \varphi_j\right). \tag{2.4}$$