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CONVERGENCE OF PARALLEL DIAGONAL ITERATION OF RUNGE-KUTTA METHODS FOR DELAY DIFFERENTIAL EQUATIONS *1)

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Abstract

Implicit Runge-Kutta method is highly accurate and stable for stiff initial value problem. But the iteration technique used to solve implicit Runge-Kutta method requires lots of computational efforts. In this paper, we extend the Parallel Diagonal Iterated Runge-Kutta(PDIRK) methods to delay differential equations(DDEs). We give the convergence region of PDIRK methods, and analyze the speed of convergence in three parts for the P-stability region of the Runge-Kutta corrector method. Finally, we analysis the speed-up factor through a numerical experiment. The results show that the PDIRK methods to DDEs are efficient.

Mathematics subject classification: 65F20.
Key words: Runge-Kutta method, Parallel iteration, Delay differential equation.

1. Introduction

We consider here a stiff initial value problem (IVP) method that is highly accurate and stable. This method is used as a corrector method, which achieves convergence by using parallel iteration techniques. In the selection of a suitable corrector method, we are automatically led to the classical implicit Runge-Kutta methods such as the Radau IIA methods. These methods fulfill the requirements of accuracy and stability and belong to the family of best correctors for stiff problems. For the iteration technique we select the PDIRK (Parallel Diagonally Implicit RK) approach developed in [1] that solves the RK corrector by diagonally implicit iteration using s processors for ODEs, where s being the number of stages of the corrector.

In this paper, we use a so-called step-parallel method. Here, a step-parallel method is understood to be a method that computes solution values at different points on the-axis simultaneously. Such methods are usually based on the iterative solution of an implicit step-by-step method. A further level of parallelism for ODE was introduced in [2,3,4] by making use of the PDIRK iteration technique. The conventional approach of iteration is that it iterates until convergence at a particular point is achieved, before advancing to the next point along the t-axis, while step-parallel methods already start the iteration process at the next point before the iteration at the preceding point converged. In the literature, we consider a step-parallel iteration of Runge-Kutta method for solving initial value problems (IVPs) of delay differential equations (DDEs):

$$y'(t) = f(y(t), y(t-\tau)), \quad t \geq 0, \quad y(t) = g(t), \quad t \leq 0, \tag{1.1}$$

where f, g denote given functions and are both sufficiently smooth. τ is a given constant with $\tau > 0$.

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2. The Iteration Scheme

Consider initial value problems of ordinary differential equations(ODEs):

$$y'(t) = f(y(t)), \quad t \geq 0, \quad y(0) = y_0. \tag{2.1}$$

To avoid the tensor product in our formulations, we consider equation(1.1) and (2.1) as scalar equations. Using the General Linear Method notation of Butcher, the Runge-Kutta corrector formula for the ODE (2.1) reads(cf.[5, 6]) as follows:

$$Y_n = EY_{n-1} + hAF(Y_n), \qquad n = 1, 2, \cdots, N, \tag{2.2}$$

here, $h(= t_n - t_{n-1})$ denotes the step-size, the matrix $A = (a_{ij})$ contains the RK parameters, and $F(Y_n)$ contains the derivative values $(f(Y_{n,i}))$, where $Y_{n,i}$, $i = 1, 2, \cdots, s$, denote components of the stage vector Y_n. In this paper we assume that (2.2) possesses s implicit stages and that the last stage corresponds to the step point. The first $s-1$ stage components represent numerical approximations at the intermediate points $t_{n-1} + c_i h$, $i = 1, 2, \cdots, s-1$, where $c = (c_i) = Ae$, $c_s = 1$, e being the vector with unit entries. We define $Y_0 = y_0 e$. The matrix E in (2.2) is of the form:

$$E = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Applying the method (2.2) to (1.1), we obtain the Runge-Kutta correction formula for DDEs:

$$Y_n = EY_{n-1} + hAF(Y_n, \gamma_n), \qquad n = 1, 2, \cdots, N. \tag{2.3}$$

Where $F(Y_n, \gamma_n)$ contains the derivative values $(f(Y_{n,i}, \gamma_{n,i}))$. If $\tau = mh$ with integer m, we let [6]

$$\gamma_n = \begin{cases} g(et_{n-1} + ch - e\tau), & n \leq m, \\ Y_{n-m}, & n > m. \end{cases} \tag{2.4}$$

We approximate the solution Y_n of (2.3),(2.4) by successive iterates $Y_n^{(j)}$ satisfying the iteration scheme:

$$\begin{aligned} &Y_n^{(0)} \text{ to be defined by the predictor formula,} \\ &Y_n^{(j)} - hDF(Y_n^{(j)}, \gamma_n) = EY_{n-1}^{(j)} + h[A - D]F(Y_n^{(j-1)}, \gamma_n), \qquad j = 1, 2, \cdots, T, \\ &\gamma_n = \begin{cases} g(et_{n-1} + ch - e\tau), & n \leq m, \\ Y_{n-m}^{(k(n)-m)}, & n > m, \end{cases} \\ &Y_n^{(j)} = Y_n^{(k(n))}, \qquad j > k(n), \qquad n = 1, 2, \cdots, N. \end{aligned} \tag{2.5}$$

The number of iterations $k(n)$ performed at the point t_n is defined by the condition that for $j = k(n)$, the $Y_n^{(j)}$s numerically satisfy the corrector equation (2.3),(2.4). The $k(n)$ depends on t_n (see(3.34)). But in a theoretical analysis, however, it seems not feasible to allow the parameter $k(n)$ to be an arbitrary function of n, so while deriving convergence results, $k(n)$ is taken as a constant. The matrix $D = (d_i)$ is assumed to be a diagonal matrix with s positive diagonal entries, so the formula (2.5) possesses parallelism across the method because of the diagonal structure of the matrix D. We also call the method (2.5) PDIRK method for DDEs.

Introducing the step index $i = n + j$, and writing the correction formula (2.5) as

$$Y_n^{(i-n)} - hDF(Y_n^{(i-n)}, Y_{n-m}^{(k)}) = EY_{n-1}^{(i-n)} + h[A - D]F(Y_n^{(i-n-1)}, Y_{n-m}^{(k)}). \tag{2.6}$$