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POLYNOMIAL PRESERVING RECOVERY FOR ANISOTROPIC AND IRREGULAR GRIDS [*1)]

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Dedicated to Professor Zhong-ci Shi on the occasion of his 70th birthday

Abstract

Some properties of a newly developed polynomial preserving gradient recovery technique are discussed. Both practical and theoretical issues are addressed. Bounded-ness property is considered especially under anisotropic grids. For even-order finite element space, an ultra-convergence property is established under translation invariant meshes; for linear element, a superconvergence result is proven for unstructured grids generated by the Delaunay triangulation.

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1. Introduction

The Zienkiewicz-Zhu error estimator [15] using recovered gradient by the superconvergent patch recovery (SPR) [16] has proven to be an effective way to access the error in computed data. The idea of their recovery is to fit higher-order polynomials, in the least-squares sense, with computed gradients on element patches. Recently, we proposed an alternative recovery method [14]. The idea is to fit higher-order polynomials with computed solution values (instead of gradient values) at some local sampling points, and obtain the recovered gradient at a nodal point by evaluating the gradient of the resultant polynomial at the same nodal point. One significant feature of this recovery is polynomial preserving. For this reason, we call it PPR.

In an earlier work [11], Wiberg-Li used function value fitting to improve convergence in the L_2-norm. In a more recent work [10], Wang used a semi-local L_2-projection and proved a superconvergent result under quasi-uniform mesh assumption.

Superconvergence properties of the SPR and its effectiveness in *a posteriori* error estimates have been studied by the author and his colleagues, see e.g., [5, 12, 13]. In this paper, we discuss PPR. Other than theoretical discussions, some practical aspects, including the selection of polynomial basis functions in the least-squares fitting and anisotropic grids are considered. Finally, we establish an ultra-convergence (two-order superconvergence) property for even-order finite elements under translation invariant meshes and a superconvergence result with irregular meshes by the Delaunay triangulation.

Numerical tests of PPR and its comparison with SPR can be found in [7, 14]. Our tests indicate that PPR is as good as, or better than SPR in practice.

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As for the literature regarding superconvergence and *a posteriori* error estimates, the reader is referred to [1, 2, 3, 4, 6, 8, 9].

2. Recovery Procedure

Let $S_{h,k}$ be a polynomial finite element space of degree k over a triangulation $\mathcal{T}_h$. We define a gradient recovery operator $G_h : S_{h,k} \to S^d_{h,k}$, with $d = 1, 2, 3$. Given a finite element solution u_h, we first define $G_h u_h$ at certain nodes. When $d = 2$, there are three types of nodes: vertices, edge nodes, and internal nodes. When $d = 3$, there is one more type: the surface node. For the linear element, all nodes are vertices. For the quadratic element, there are vertices and edge-center nodes. For the cubic or higher-order elements, all types of nodes are present. After defining values of $G_h u_h$ at all nodes, we obtain $G_h u_h \in S^d_{h,k}$ on the whole domain by interpolation using the original nodal shape functions of $S_{h,k}$.

Given a node $\boldsymbol{z}_i$, we need to determine $G_h u_h(\boldsymbol{z}_i)$. This is achieved by first selecting $n \geq m = \frac{1}{d!}\Pi_{j=1}^d (k+1+j)$ sampling points adjacent to $\boldsymbol{z}_i$ (including $\boldsymbol{z}_i$), and then fitting a polynomial of degree $k+1$, in the least-squares sense, with values of u_h at those sampling points. In other words, we are looking for $p_{k+1} \in \mathcal{P}_{k+1}$ such that

$$\sum_{j=1}^{n} (p_{k+1} - u_h)^2(\boldsymbol{z}_{ij}) = \min_{q \in \mathcal{P}_{k+1}} \sum_{j=1}^{n} (q - u_h)^2(\boldsymbol{z}_{ij}). \tag{2.1}$$

Using the local coordinates (x, y) with $\boldsymbol{z}_i$ as the origin, the fitting polynomial is denoted as $p_{k+1}(x, y; \boldsymbol{z}_i)$, we then define

$$G_h u_h(\boldsymbol{z}_i) = \nabla p_{k+1}(0, 0; \boldsymbol{z}_i). \tag{2.2}$$

Comparing with Zienkiewicz-Zhu's patch recovery [16], here we fit u_h instead of ∇u_h. The above procedure generates a finite difference scheme

$$G_h v(\boldsymbol{z}_i) = \sum_{j=1}^{n} \vec{C}_j v(\boldsymbol{z}_{ij}), \quad \sum_{j=1}^{n} \vec{C}_j = \vec{0}. \tag{2.3}$$

The task now is to determine the coefficients $\vec{C}_j$s.

Usually, we select sampling points as nodal points of all triangles that share a common vertex $\boldsymbol{z}_i$. These triangles naturally form an element patch as used in [16]. Figures 4-6 depict some possible interior and boundary patches when $d = 2$. Among them, only the last two interior patches (with 4 and 5 triangles, respectively) in Figure 4 and the two boundary patches in Figure 5 appear in meshes constructed by a sophisticated automatic mesh generator (based on the Delaunay triangulation). Indeed, for an interior patch that has only three triangles (the first patch in Figure 4), a mesh generator simply removes the center node and three connecting edges; for a patch that contains four triangles (the second patch in Figure 4), a mesh generator removes the center node and related edges, then adds one of the diagonals of the quadrilateral to form two new triangles. As for a boundary vertex, a mesh generator always seeks to connect it with two interior vertices, and for a corner vertex, a mesh generator always bisects the angle that is less than $\pi/2$. Therefore, situations in Figure 6 and the first two cases in Figure 4 almost never happen in practice.

Actually, the sampling points selection can be very flexible. The rule of thumb is to make an interior node $\boldsymbol{z}_i$ as close as possible to the geometric center of all $\boldsymbol{z}_{ij}$s. The perfect situation is when $\boldsymbol{z}_{ij}$s are symmetrically distributed around $\boldsymbol{z}_i$.

We may always select $n \geq m$ sampling points. However, this alone is not sufficient to guarantee that problem (2.1) has a unique solution. Towards this end, we introduce an **Angle condition**: The sum of any two adjacent angles in $\mathcal{T}_h$ is no more than π.

Theorem 1. *The angle condition implies a unique solution of (2.1) when $n \geq m$.*