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RECIPROCAL POLYNOMIAL EXTRAPOLATION [*1)]

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Abstract

An alternative to the classical extrapolations is proposed. The stability and the accuracy are studied. The new extrapolation behaves better than the classical ones when there are problems of stability. This technique will be useful in those problems where the region of stability is very small and it forces to work with too fine scales.

Key words: Extrapolation, Stability, ODE's.

1. Introduction

In order to improve the accuracy of a numerical method to approximate any quantity, several strategies are used, in most cases increasing its computational cost. Thus, reducing the discretization parameters, or obtaining higher order methods get better resolution, but the number of arithmetic computations to do is drastically increased. One of the best known strategies to get higher order methods based on a given one is extrapolation.

The discretization in space of PDE's leads to stiff systems of ODE's. When we apply the classical Runge-Kutta methods to these systems an order reduction takes place. A possibility is to use extrapolation methods in order to increase the final order. Moreover, we need to have a method with good stability properties. We are going to present a new extrapolation verifying these properties.

Let X and Y Banach spaces and $A : D(A) \subset X \to Y$ a linear operator. We consider the problem:

$$\frac{d}{dt}x(t) = Ax(t) + f(t), \; 0 \leq t \leq T, \tag{1}$$

$$x(0) = u_0 \in X, \tag{2}$$

where $T > 0$.

If we use a Runge-Kutta method to solve this problem we will have an approximation in the way:

$$S(h) = a_0 + a_1 h^p + \dots \tag{3}$$

where a_0 is the exact solution. Then considering, for example, two different discretizations h and $2h$, an extrapolation methods make a new approximation with error $O(h^r), \; r > p$.

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In this paper we carry out a theoretical study (analyzing the errors, the regions of stability...) and later, one practical. We will present a comparison with the classical rational and polynomial extrapolations.

The structure of the paper is the following: In section two we introduce the extrapolation concept, in section three we present the reciprocal polynomial extrapolation, the error and the stability are analyzed in section four, finally, we present some numerical results and conclusions in section five.

2. The Extrapolation Concept

Usually, if we assume we are approximating a value a_0 by means of a numerical method, we get the approximation $S(h)$, where

$$\lim_{h\to 0} S(h) = a_0. \tag{4}$$

The error of classical methods can be expressed as a Taylor series:

$$S(h) = a_0 + a_1 h^{\gamma_1} + a_2 h^{\gamma_2} + \dots \tag{5}$$

The most classical extrapolation method consists of successive elimination of the terms $a_i h^{\gamma_i}$ by linear combinations of approximations $S(h)$ for different h (Richardson's extrapolation). This can be viewed as the value of the only polynomial $P(x)$ interpolating those chosen $S(h)$ in $h = 0$.

Nevertheless, we can consider not only linear but also nonlinear strategies. We say $\phi(h, h^{'})$ is an extrapolation of (5) if

$$\phi(h, h^{'}) = a_0 + a_2^{'} h^{\hat{\gamma}_1} + \dots \tag{6}$$

with $\hat{\gamma}_1 > \gamma_1$.

It is clear, the order of accuracy is the same both cases. Our goal is to analyze the stability concept (in stiff problem nonlinear extrapolation obtain better practical results). The most used nonlinear extrapolation is a special class of rational one, in which the degrees of the polynomials in the rational function follow a special sequence. This extrapolation can be use when the exponents of the error expansion follow an arithmetic sequence only (see [7] for more details). On the other hand, in stiff problems, implicit methods are used, because they have better properties of stability. In this case we have to solve a nonlinear equation associated to the implicit method. The nonlinear equation can be to solve with an error $O(h^{p+1})$, where p is the order of the original methods. Thus, if we consider a coarser discretization, like extrapolation methods, then we can solve the equation with a bigger error.

In this paper we will study the behavior of $\frac{1}{P(x)}$ (reciprocal polynomial extrapolation). We will see all the classical methods can be extrapolated by this technique, and it will have better stability properties than usual rational and polynomial extrapolations.

3. Interpretation of the Reciprocal Polynomial Extrapolation

Richardson's extrapolation is equivalent to extrapolate by couples with functions of type $ax^p + b$ (p =order of the method in each step). In our case, it will be equivalent to consider the extrapolator function $R(x) = \frac{1}{P(x)}$ than considering extrapolations by couples with functions of the type $\frac{1}{dx^p+e}$.

Notice you in first place that for the construction of $\frac{1}{dx^p+e}$, one has to suppose that S_{h_i} and $S_{h_{i+1}}$ have the same sign. If $S_{h_i} S_{h_{i+1}} \leq 0$, then, to find the rational function, first we would make a translation of the axis, we would build then $\frac{1}{dx^p+e}$ and we would make the inverse translation lastly.