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A COMPUTATIONAL APPROACH FOR OPTIMAL CONTROL SYSTEMS GOVERNED BY PARABOLIC VARIATIONAL INEQUALITIES *

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Abstract

Iterative techniques for solving optimal control systems governed by parabolic variational inequalities are presented. The techniques we use are based on linear finite elements method to approximate the state equations and nonlinear conjugate gradient methods to solve the discrete optimal control problem. Convergence results and numerical experiments are presented.

Key words: Parabolic variational inequalities, finite elements method, nonlinear conjugate gradient methods.

1. Introduction

In the theory of variational inequalities and their approximation by finite elements methods the dam problem models hold a particular place (see for example [3], [14], and references therein). Such models are of a great practical interest in the development and management of water resources. Knowledge of the amount of seepage is essential for water conservation practice. The main goal of the present paper is to study numerical approximations for the following optimal control problem (CP):

$$\begin{cases} \text{Find} \quad q_{min} \in U_{ad} \subset L^2(0,T) \quad \text{such that} \\ J_j(q_{min}) = \min_{\forall q \in U_{ad}} J_j(q), \quad j = 1,2,...,m, \end{cases} \tag{1.1}$$

subjected to

$$\dot{H}_{j-1}(t) + \tilde{\alpha} H_{j-1}(t) = q(t), \quad j = 1,2,...,m. \tag{1.2}$$

The cost functionals are defined by

$$J_j(q) := \frac{1}{2}\int_{Q_j} [w_j(q) - w_j^d]^2 \, dx \, dt + \frac{N}{2}\int_0^T [q]^2 \, dt, \quad j = 1,2,...,m, \tag{1.3}$$

where N is a nonnegative real. Let us denote by $q(t)$ the control variable, by U_{ad} a closed convex subset in $L^2(0,T)$ and by $w_j^d(x,t)$ a given functions in $L^2(Q_j)$. Also this problem subjected to the following parabolic system of variational inequalities (Problem (P)):

$$\begin{cases} w_j(t) \geq 0 \ a.e., w_j \in L^2(0,T;H^2(D_j)), \ \dfrac{\partial w_j}{\partial t} \in L^2(0,T;H^1(D_j)), \ (\text{then}\, w_j \in C^0(\bar{Q}_j)\,), \\ \nu\,(\dfrac{\partial w_j}{\partial t}, v_j - w_j(t))_{L^2(D_j)} + a(w_j(t), v_j - w_j(t)) \geq -(1, v_j - w_j(t))_{L^2(D_j)}, \\ \forall v_j \in L^2(D_j), \, a.e., \quad w_j(t_{j-1}) = \tilde{w}_{j-1}, \quad w_j(t) \in K_j(t), \quad v_j \geq 0 \quad j = 1,2,...m, \end{cases}$$

where

$$K_j(t) = \{v_j \in H^1(D_j) : v_j = G_j \text{ on } \Gamma_{dj}, \, j = 1,2,...,m\},$$

G_j are any functions in $H^2(Q_j)$ such that the value of G_j on the boundary Γ_{n_j}, $\hat{g}_j$, is assumed to have a zero derivative and $G_j = g_j$ on $\Gamma_{d_j} \times]t_{j-1}, T[$ such that $g_j \geq 0$, $j = 1,2,...,m$,

$$\begin{cases} g_j(a_j, y, t_j) := \displaystyle\int_{t_{j-1}}^{t_j} [H_{j-1}(\tau) - (y + t_j - \tau)]^+ d\tau + \frac{1}{2}[(H_{j-1}(t_{j-1}) - y - t_j)^+]^2 \,, \\ g_j(b_j, y, t_j) := \displaystyle\int_{t_{j-1}}^{t_j} [H_j(\tau) - (y + t_j - \tau)]^+ d\tau + \frac{1}{2}[(H_j(t_j) - y - t_j)^+]^2 \,, \\ g_j(x, e_j, t_j) := 0, \quad m^+ := \dfrac{(|m| + m)}{2}, \end{cases}$$

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and $a(u,v) := (\bigtriangledown u, \bigtriangledown v)_{L^2(D_j)}$ is the bilinear, coercive form in $H^1(D_j)$.

Physically, the above model describes (see [27]) the optimal control evolution system of earth dams where we consider an unsteady flow, say water, moving through **m** homogeneous porous rectangular earth dams. The dams have the following domains $D_j := \{(x,y) \mid 0 < a_j < x < b_j,\ 0 < y < e_j\ ,\ j = 1,2,...,m\}$, respectively, with vertical walls $x = a_j,\ x = b_j,\ \ j = 1,2,...,m$. We suppose that the water levels $H_0(t), H_m(t)$ are given real numbers, $H_0(t) > H_m(t) > 0$ and $H_j(t)$ are the intermediate water levels between j^{th} and $(j+1)^{\text{th}}$ dams, $j = 1,2,...,m-1$ (see Figure 1), $w_j(q)$ are the weak solutions of the evolution dams problem when the initial data follows from the stationary dam problem (see [6],[10],[28]) and $\tilde{w}_{j-1}$ are the solutions of the stationary dams problem (see [3]). The constant $\nu \geq 0$ is called the retentivity coefficient, the case $\nu = 0$ is related to an incompressible fluid. We will denote by $t_j \in [0,T], 0 < T < +\infty$ the time intervals during which we want to study the filtration dams, $j = 1,2,...,m$, $t_0 = 0$. Let $Q_j = D_j \times (t_{j-1}, t_j),\ j = 1,2,...,m,$

$$\Gamma_{nj} = \{(x,y) : a_j < x < b_j, y = 0\}, \quad \Gamma_{d_j} := \Gamma_j - \Gamma_{nj}.$$

where Γ_j are the smooth boundary of D_j. Optimal control problems in connection with variational inequalities contain many difficulties, e.g., [4], [5], [12], [16], [17], [18] and [19] or more recently [1], [7] and the references therein. The control problem $(1.1)-(1.3)$ is in general a non-convex and non-differentiable optimization problem, see [18]. It has been proved in ([27]) that by controlling the amount of fluid that may go out of each dam the free boundary in each dam can be controlled. Also in ([27]) regularizing the problem necessary optimality conditions were exhibited and obtained convergence results when the regularization parameter tends to zero. One justification to use these methods for solving variational inequalities numerically is the fact that inequalities are replaced by equations (see for example [23] and [26]). For simplicity we will write w_j instead of $w_j(q)$. Also we write

$$g_j(q) := g_j(a_j, y, t_j)\,, \quad \forall y \in D_j,\, t_j \in [0,T]\,, \quad j = 1,2,...,m \tag{1.4}$$

which is continuous functions. This under the assumptions $|H_j(t)| \leq \tilde{C}_1, j = 1,3,5,...,m$, we have $|g_j| \leq \tilde{C}_2$ on the sides $x = b_j$ where $\tilde{C}_1$ and $\tilde{C}_2$ are posstive constant. In the sequel we do not care about the existence of an optimal solution of (1.1)-(1.3): one can refer to [27], where the above formulations and the following theorems in this section can be found.

Problem (P^ε). *Consider the ε-approximating problem for (P) as follows:*

$$\begin{cases} \forall \varepsilon > 0,\ \forall v \in V_j \quad \text{find the functions}\, w_j^\varepsilon(q_j) \ \text{ such that,} \\ \nu\left(\dfrac{\partial w_j^\varepsilon}{\partial t}, v\right) + a\left(w_j^\varepsilon, v\right) + \dfrac{1}{\varepsilon}\left(\beta(w_j^\varepsilon, v\right) = \left(1, v\right) \quad a.e.,\ \ t \in [0,T]\,, \\ w_j^\varepsilon(t_{j-1}) = \tilde{w}_{j-1}^\varepsilon, \quad w_j^\varepsilon(q_j)|_{\Gamma_{d_j}} = g_j^\varepsilon(q_j)\,, \end{cases}$$

where $V_j = \{v \in H^1(D_j) : v = 0 \text{ on } \Gamma_{d_j}\}$, $g_j^\varepsilon(q)$ is a regularized function for $g_j(q)$, see [17] for example we can choose $g_j(q)$, as an exterior penalized function in $H^2(D_j \times (0,T))$. We can prove (iteratively) that this problem has a unique solution in $L^2(0,T,V_j) \cap H^1(0,T,V_j')$ (see Barbu [4] pp. 160) where V_j' is the dual of V_j. To emphasize the penalization method, therefore for any $v \in H^1(0,T;V)$ the metric projection is given by (see R. Scholz [23])

$$P(v) := v - v^+, \tag{1.5}$$

therefore we introduce a penalty function $\beta(v)$,

$$\beta(v) := v - P(v) = v^+, \tag{1.6}$$

$$\beta'(v(t)) = \begin{cases} v'(t) = \frac{d}{dt}v(t) & \text{for } \beta(v) \neq 0, \\ 0 & otherwise, \end{cases} \quad \Big(v'(t), \beta(v)\Big) = \frac{1}{2}\frac{\partial}{\partial t}\,\|\beta(v)\|^2_{L^2(D)}. \tag{1.7}$$

Lemma 1. *Let w_j^ε be the solutions of (P^ε), $\varepsilon > 0$, $j = 1,2,...,m$. Then the estimates*

$$\|\beta(w_j^\varepsilon)\|_{L^2(Q_j)} \leq C\varepsilon, \qquad \|\beta(w_j^\varepsilon)\|_{L^2(0,T,H^1(D_j))} \leq C\varepsilon^{\frac{1}{2}}, \tag{1.8}$$

$$\|w_j^\varepsilon\|_{L^\infty(0,T,L^2(D_j))} + \|w_j^\varepsilon\|_{L^2(0,T,V)} + \|\frac{\partial w_j^\varepsilon}{\partial t}\|_{L^2(0,T,V')} \leq C\,\|g^\varepsilon{}_j\|_{L^2(0,T,H^{\frac{1}{2}}(\Gamma_{d_j}))},$$

are hold and C is independent of ε.