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DISSIPATIVITY AND EXPONENTIAL STABILITY OF $\theta-$METHODS FOR SINGULARLY PERTURBED DELAY DIFFERENTIAL EQUATIONS WITH A BOUNDED LAG *1)

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Abstract

This paper deals with analytic and numerical dissipativity and exponential stability of singularly perturbed delay differential equations with any bounded state-independent lag. Sufficient conditions will be presented to ensure that any solution of the singularly perturbed delay differential equations (DDEs) with a bounded lag is dissipative and exponentially stable uniformly for sufficiently small $\varepsilon > 0$. We will study the numerical solution defined by the linear $\theta-$method and one-leg method and show that they are dissipative and exponentially stable uniformly for sufficiently small $\varepsilon > 0$ if and only if $\theta = 1$.

Key words: Singular perturbation, $\theta-$methods, Dissipativity, Exponential stability.

1. Introduction

Singular perturbation problems(SPPs) form a special class of problems containing a small parameter ε. They are of practical interest in models of instantaneous phenomena and include a subclass of what we frequently thought of as 'stiff' equations. Singularly perturbed delay differential equations of the form

$$\varepsilon y'(t,\varepsilon) = g(t, y(t,\varepsilon), y(t-\tau,\varepsilon)), \qquad 0 \le t \le T, \tag{1}$$

subject to the initial condition

$$y(t,\varepsilon) = \phi(t,\varepsilon), \qquad -\tau \le t \le 0 \tag{2}$$

arise in the study of an "optically bistable device" [7] and in a variety of models for physiological processes or diseases [16]. Such a problem has also appeared to describe the so-called human pupil-light reflex [15]. For example, Ikeda [13] adopted the model

$$\varepsilon y'(t,\varepsilon) = -y(t,\varepsilon) + A^2\left[1 + 2B\cos(y(t-1,\varepsilon))\right]$$

to describe an optically bistable device and showed numerically that instability or chaotic behaviour occurs for small ε and certain values of A, B. This was confirmed experimentally by Gibbs, Hopf, Kaplan and Shoemaker [9].

1.1. A Simple Example

Before we investigate dissipativity and exponential stability of singularly perturbed delay differential equations, we first consider a simple ordinary differential equation in the form

$$\begin{aligned} \varepsilon y'(t) &= \lambda y(t), \quad (\Re\lambda \le 0), \quad t > 0, \\ y(0) &= y_0, \end{aligned} \tag{3}$$

which has the solution

$$y(t) = e^{\frac{\lambda}{\varepsilon}t} y_0.$$

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The most obvious classical difference scheme for solving this problem numerically is $\theta-$method:

$$\varepsilon(y_{n+1} - y_n) = \theta\lambda h y_{n+1} + (1-\theta)\lambda h y_n, \tag{4}$$

where $n \geq 0$. Solving it explicitly, we obtain

$$y_{n+1} = \frac{\varepsilon + (1-\theta)\lambda h}{\varepsilon - \theta\lambda h} y_n, \tag{5}$$

in which $\frac{\varepsilon+(1-\theta)\lambda h}{\varepsilon-\theta\lambda h}$should be an approximation to $e^{\frac{\lambda}{\varepsilon}h}$.

There are several disadvantages of the $\theta-$method. First, the $\theta-$method doesn't possess uniform convergence in ε. Let $\rho = \frac{h}{\varepsilon}$. The general form of the first mesh error is

$$\lim_{h\to 0} |y(h) - y_1| = \left| e^{\lambda\rho} - \left(1 + \frac{\lambda\rho}{1-\theta\lambda\rho}\right)\right| |y_0|. \tag{6}$$

When $\rho = 1, y_0 \neq 0$, for example, (6) reads

$$\lim_{h\to 0} |y(h) - y_1| = \left| e^{\lambda} - \left(1 + \frac{\lambda}{1-\theta\lambda}\right)\right| |y_0| \neq 0, \tag{7}$$

which means nonuniform convergence in ε. In addition, it can be proved that

$$\lim_{\substack{h \to 0 \\ \rho \to \infty}} |y(h) - y_1| = 0 \tag{8}$$

for any initial value if and only if $\theta = 1$.

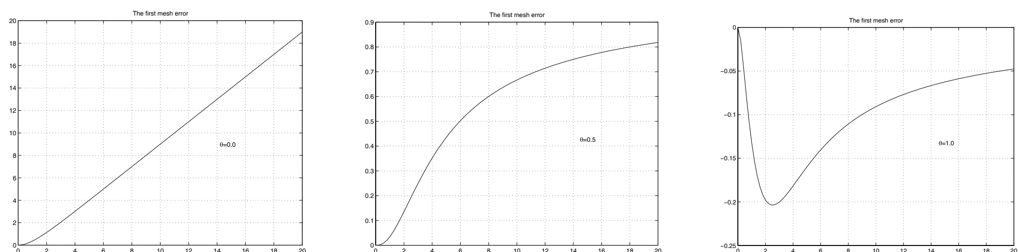

Figure 1: Graph of error function with respect to the ratio ρ of stiff coefficient ε and step-size h.

Figure 1 implies that the error is small only for small ρ when $\theta \neq 1$, while for the backward Euler method (i.e. $\theta = 1$), the error is small for small and large ratio ρ and becomes significant when ε and h are of the same order of magnitude.

Second, it is clear that the discrete solution oscillates if $\rho > \frac{1}{-\lambda}$ (where $\lambda \in \mathbb{R}$) except $\theta = 1$, because

$$y_n = \left(1 + \frac{\lambda\rho}{1-\theta\lambda\rho}\right)^n y_0. \tag{9}$$

These oscillations are spurious since they do not occur in the solution of the continuous problem, and can only be avoided by taking the backward Euler scheme.

Third, the original equation is asymptotically stable and hence numerical approximation should mimic the same property, which requires

$$\left|\frac{\varepsilon + (1-\theta)\lambda h}{\varepsilon - \theta\lambda h}\right| < 1. \tag{10}$$

It is well-known that $\theta-$method is A-stable for ODEs if and only if $\theta \in [\frac{1}{2}, 1]$. Unfortunately, it is easy to verify that (10) is satisfied for any $\lambda h \in \{x : \Re x < 0\}$ uniformly in $\varepsilon > 0$ if and only if $\theta \in (\frac{1}{2}, 1]$, which rules out the trapezoidal method since it is not strongly stable at infinity.

We distinguish two cases: