Journal of Computational Mathematics, Vol.21, No.4, 2003, 421–434.

GAUSS-SEIDEL-TYPE MULTIGRID METHODS*[1)]

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Abstract

By making use of the Gauss-Seidel-type solution method, the procedure for computing the interpolation operator of multigrid methods is simplified. This leads to a saving of computational time. Three new kinds of interpolation formulae are obtained by adopting different approximate methods, to try to enhance the accuracy of the interpolatory operator. A theoretical study proves the two-level convergence of these Gauss-Seidel-type MG methods. A series of numerical experiments is presented to evaluate the relative performance of the methods with respect to the convergence factor, CPU-time(for one V-cycle and the setup phase) and computational complexity.

Key words: Multigrid methods, Gauss-Seidel solution, Interpolation formula, Convergence.

1. Introduction

Multigrid(MG) methods are very efficient methods for solving linear systems with a broad range of applications. The characteristic feature of multigrid iteration is its fast convergence. This convergence speed does not deteriorate when the discretization is refined, unlike for classical iterative methods which slow down for decreasing grid size. As a consequence an acceptable approximation of the discrete problem can be obtained at the expense of computational work proportional to the number of unknowns, which is also the number of equations in the system. It is not only the complexity which is optimal, but also the constant of proportionality is so small that other methods can rarely surpass multigrid efficiency [14] [2] [12].

Usual multigrid methods try to tailor the components to the problem at hand in order to obtain the highest possible efficiency for the solution process. However, the algebraic multigrid(AMG) method is to choose the components independently of the given problem, uniformly for as large a class of problem as possible. AMG provides a very robust solution method which can be applied directly to structured as well as unstructured grids. The strengths of AMG are exactly its robustness, its applicability in complex geometric situations and its applicability to even solve certain problems which are out of the reach of usual multigrid methods, in particular, problems with no geometric or continuous background at all as long as the given matrix satisfies certain conditions [1] [13].

There now exist various different algebraic approaches, all of which are hierarchical and close to the original AMG idea, but some of which focus on different coarsening and interpolation procedures. An efficient AMG algorithm for M-matrices is described in [13]. In [3] [4] [5] [6] [7] [8] [9] and [10], Chang et al. improve the interpolation operator and present different algorithms to construct the coarse grid equations. In this paper, we apply the Gauss-Seidel solution method to the computation of operator-dependent interpolation, and thus suggest three new Gauss-Seidel-type multigrid methods, whose convergence is also proved. Furthermore, many numerical examples are discussed to compare the efficiency of AMG methods. In section 2, the basic AMG algorithm by Chang is described. Then three different AMG methods are proposed

* Received January 15, 2001.
1) This work is supported in part by a grant (No. 19931030) from the National Natural Science Foundation of China.

in section 3. In section 4, a convergence analysis for each of the AMG algorithms is given. Numerical analysis and computational results are reported in section 5. Finally, conclusions are presented in section 6.

2. The Basic AMG Algorithm

Consider a (sparse) linear system of equations: $AU = F$ or $\sum_{j=1}^{n} a_{ij}u_j = f_i (i = 1, \cdots, n)$. We first have to generate a sequence of smaller and smaller systems of equations: $A^m U^m = F^m$ or $\sum_{j=1}^{n_m} a_{ij}^m u_j^m = f_i^m (i = 1, \cdots, n_m)$, where $A^m = (a_{ij}^m)_{n_m \times n_m}, U^m = (u_1^m, \cdots, u_{n_m}^m)^T, F^m = (f_1^m, \cdots, f_{n_m}^m)^T$, $m = 1, \cdots, M, n = n_1 > \cdots > n_M, A^1 = A, U^1 = U, F^1 = F$.

A fictitious grid Ω^m can be regarded as a set of unknown $u_j^m (1 \le j \le n_m)$. The coarser grid Ω^{m+1} is then chosen as a subset of Ω^m, which is denoted by C^m. The remainder subset $\Omega^m - C^m$ is denoted by F^m. A point i is said to be strongly connected to j, if

$$|a_{ij}^m| \ge \theta \cdot \max_{k \ne i} |a_{ik}^m|, 0 < \theta \le 1.$$

Let S_i^m denote the set of all atrongly connected points of the point i and let $C_i^m = C^m \bigcap S_i^m, N_i^m = \{j \in \Omega^m, j \ne i, a_{ij}^m \ne 0\}, D_i^m = N_i^m - C_i^m, D_i^s = D_i^m \bigcap S_i^m, D_i^w = D_i^m - D_i^s$.

Each variable in C^m interpolates directly from the corresponding variable in Ω^{m+1} with a weighting of unity, and each variable $i \in F^m$ interpolates from the smaller set C_i^m.

Because the error e_i^m to be interpolated in an AMG method is obtained after a smoothing process, we have

$$a_{ii}^m e_i^m + \sum_{j \in N_i^m} a_{ij}^m e_j^m = d_i^m \approx 0, \forall i \in F^m, \tag{2.1}$$

which can be rewritten as

$$a_{ii}^m e_i^m + \sum_{k \in C_i^m} a_{ik}^m e_k^m + \sum_{j \in D_i^s} a_{ij}^m e_j^m + \sum_{j \in D_i^w} a_{ij}^m e_j^m \approx 0, \forall i \in F^m. \tag{2.2}$$

Let $g_{jk}^m = \frac{|a_{jk}^m|}{\sum_{k \in C_i^m} |a_{jk}^m|}, j \in D_i^m, k \in C_i^m$, for point $j \in D_i^m$, the following approximations are used in (2.2):

(1) For point $j \in D_i^w$,

$$e_j^m = \begin{cases} e_i^m, & \text{if } l_{ij}^m = 0, a_{ij}^m < 0, \\ -e_i^m, & \text{if } l_{ij}^m = 0, a_{ij}^m > 0, \\ 2\sum_{k \in C_i^m} g_{jk}^m e_k^m - e_i^m, & \text{if } l_{ij}^m > 0, \xi_{ij}^m \ge 0.5, a_{ij}^m < 0, \\ \sum_{k \in C_i^m} g_{jk}^m e_k^m, & otherwise. \end{cases} \tag{2.3}$$

(2) For points $j \in D_i^s$, more accurate approximations are used,

$$e_j^m = \begin{cases} 2\sum_{k \in C_i^m} g_{jk}^m e_k^m - e_i^m, & \text{if } \eta_{ij}^m < 0.75, \xi_{ij}^m \ge 0.5, a_{ij}^m < 0, \\ \frac{1}{2}(\sum_{k \in C_i^m} g_{jk}^m e_k^m + e_i^m), & \text{if } \eta_{ij}^m > 2, \xi_{ij}^m \ge 0.5, a_{ij}^m < 0, \\ \sum_{k \in C_i^m} g_{jk}^m e_k^m, & otherwise. \end{cases} \tag{2.4}$$

where

$$\xi_{ij}^m = \frac{\sum_{k \in C_i^m} a_{jk}^m}{\sum_{k \in C_i^m} |a_{jk}^m|}, \eta_{ij}^m = \frac{|a_{ji}^m| l_{ij}^m}{\sum_{k \in C_i^m} |a_{jk}^m|}, l_{ij}^m = |S_{ij}^m|, S_{ij}^m = \{k : k \in C_i^m, a_{jk}^m \ne 0\}.$$

Substituting (2.3)-(2.4) into (2.2) is equivalent to modifying the coefficients in (2.2) by combining the following steps:

Step(1) add $-|a_{ij}^m|$ to $a_{ii}^m, \forall j \in D_i^{(1)}$, which is equivalent to e_j^m being replaced by e_i^m or $-e_i^m$;

Step(2) add $a_{ij}^m g_{jk}^m$ to $a_{ik}^m, \forall k \in C_i^m, \forall j \in D_i^{(2)}$, which is equivalent to e_j^m being approximated