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MULTIGRID FOR THE MORTAR FINITE ELEMENT FOR PARABOLIC PROBLEM *1)

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Abstract

In this paper, a mortar finite element method for parabolic problem is presented. Multigrid method is used for solving the resulting discrete system. It is shown that the multigrid method is optimal, i.e, the convergence rate is independent of the mesh size L and the time step parameter τ.

Key words: Multigrid, Mortar element, Parabolic problem.

1. Introduction

The mortar finite element is a new type of domain decomposition method, which can handle the situations where subdomain meshes may be separately constructed and nonmatching along the interface. We refer the reader for the general presentation of the mortar element method to [3]. In [1], some domain decomposition preconditioners were constructed for the discrete system of the mortar element method. Recently, a variable V-cycle multigrid preconditioner and a W-cycle multigrid for the mortar element method were presented in [7],[4].

The objective of this paper is to study the mortar finite element for parabolic problem. First, we extend the results in [3] to parabolic problem. An optimal energy error is obtained. Meanwhile, we consider a multigrid method for solving the discrete system resulting from the mortar finite element method. It is shown that the multigrid method is optimal, i.e., the convergence rate is independent of the mesh size L and the time step parameter τ.

2. Parabolic Problem

Consider the following parabolic problem: to find $u(x,t)$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{L}u &= f \quad in \quad \Omega\times[0,T], \\ u(x,t) &= 0 \quad in \quad \partial\Omega\times[0,T], \\ u(x,0) &= u_0(x), \end{cases} \tag{2.1}$$

where $\Omega\subset R^2$ is a bounded domain, $f\in L^2(\Omega)$. $\mathcal{L}$ is an elliptic operator

$$\mathcal{L}u = -\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}). \tag{2.2}$$

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Here $a_{ij}(x)$ satisfies

$$c\xi^t\xi \le \sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \le C\xi^t\xi \quad \forall x \in \Omega, \xi \in R^d, \tag{2.3}$$

where c,C are positive constants.

The variational form of (2.1) is to find $u \in H_0^1(\Omega)$, $u(x,0) = u_0(x)$ such that

$$(\frac{\partial u}{\partial t}, v) + B(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega), \quad t \in [0,T], \tag{2.4}$$

where the bilinear form B is

$$B(u,v) = \int_\Omega \sum_{i,j=1}^{d} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}dx \quad \forall u,v \in H^1(\Omega)$$

and

$$(f,v) = \int_\Omega fvdx.$$

We refer the notations of Sobolev space to [6] for details. It is easily seen that the bilinear form $B(u,v)$ is

(1). bounded, i.e. $|B(u,v)| \le C|u|_1|v|_1 \quad \forall u,v \in H_0^1(\Omega)$.

(2). elliptic, i.e. $|B(u,u)| \ge C|u|_1^2 \quad \forall u \in H_0^1(\Omega)$.

We use the backward Euler scheme and Crank-Nicolson scheme for the time discretization [10]. Both schemes are absolutely stable [8]. Let $\triangle t_n$ be the n^{th} time step and M_1 the number of steps, then $\sum_{n=1}^{M_1} \triangle t_n = T$. We lead to the following problem: for a given function $g_{n-1} \in L^2(\Omega)$, find $w \in H_0^1(\Omega)$ such that

$$A_\tau(w,v) = \tau^{-1}(w,v) + B(w,v) = (g_{n-1},v) \quad \forall v \in H_0^1(\Omega), \tag{2.5}$$

where τ is the time step parameter. For the backward Euler scheme, we have

$$\begin{aligned} &w = u^n - u^{n-1}, \\ &\tau = \triangle t_n, \\ &(g_{n-1},v) = (f,v) - B(u^{n-1},v), \end{aligned}$$

and for the Crank-Nicolson scheme, we have

$$\begin{aligned} &w = u^n - u^{n-1}, \\ &\tau = \triangle t_n/2, \\ &(g_{n-1},v) = 2((f,v) - B(u^{n-1},v)). \end{aligned}$$

It is known [6] that if Ω is a convex polygon, then for any $g \in L^2(\Omega)$, there exists a solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ of

$$B(u,v) = (g,v), \quad \forall v \in H_0^1(\Omega) \tag{2.6}$$

with

$$\|u\|_2 \le C\|g\|_0. \tag{2.7}$$

Here and throughout this paper, c and C (with or without subscript) denote generic positive constants, independent of the time step parameter τ, the mesh parameters L and h_L which will be stated below.

Based on the regularity assumption (2.7), we have

Lemma 2.1. *For any $g \in L^2(\Omega)$, the equation*

$$A_\tau(u,v) = (g,v) \quad \forall v \in H_0^1(\Omega) \tag{2.8}$$