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JACOBI SPECTRAL METHODS FOR MULTIPLE-DIMENSIONAL SINGULAR DIFFERENTIAL EQUATIONS*[1)]

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Abstract

Jacobi polynomial approximations in multiple dimensions are investigated. They are applied to numerical solutions of singular differential equations. The convergence analysis and numerical results show their advantages.

Key words: Jacobi approximations, Multiple dimensions.

1. Introduction

The spectral method has high accuracy. However, it might be destroyed by the singularity of genuine solutions. Guo [1], and Guo and Wang [2] developed Jacobi approximations to singular differential equations. But so far, there is no work in multiple dimensions. This paper is devoted to Jacobi spectral method for multiple-dimensional singular differential equations. We first recall some basic results on Jacobi approximation, and then give the main results of this paper. They are used for numerical solutions of singular differential equations. The convergence analysis and numerical results show the efficiency of this approach.

2. Some Basic Results on Jacobi Approximations

Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain, $x \in \mathbb{R}^d$ and $\chi(x)$ be certain weight function. We define the weighted space $L^p_\chi(\Omega)$ and its norm $||v||_{L^p_\chi}$ in the usual way. Denote the inner product and the norm of the space $L^2_\chi(\Omega)$ by $(u,v)_\chi$ and $||v||_\chi$. We define the weighted Sobolev space $H^r_\chi(\Omega)$ as usual with the inner product $(u,v)_{r,\chi}$, the semi-norm $|v|_{r,\chi}$ and the norm $||v||_{r,\chi}$.

We recall some basic results on the Jacobi approximations. Let $d=1, \Omega \equiv \Lambda = (-1,1)$ and $\chi^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$. For $\alpha, \beta > -1$,

$$(J_l^{(\alpha,\beta)}, J_m^{(\alpha,\beta)})_{\chi^{(\alpha,\beta)}} = \gamma_l^{(\alpha,\beta)}\delta_{l,m}, \qquad \gamma_l^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(l+\alpha+1)\Gamma(l+\beta+1)}{(2l+\alpha+\beta+1)\Gamma(l+1)\Gamma(l+\alpha+\beta+1)}. \tag{2.1}$$

Let $\mathbb{N}$ be the set of all non-negative integers. For any $N \in \mathbb{N}$, $\mathcal{P}_N$ stands for the set of all algebraic polynomials of degree at most N. Further let ${}_0\mathcal{P}_N = \{v | v \in \mathcal{P}_N, v(-1) = 0\}$ and $\mathcal{P}^0_N = \{v | v \in \mathcal{P}_N, v(-1) = v(1) = 0\}$. Denote by c a generic positive constant independent of any function and N.

Lemma 2.1. (Lemma 3.7 of [2] and Lemma 2.4 of [1]). *If* $-1 < \alpha, \beta < 1$, *then for any* $v \in H^1_{0,\chi^{(\alpha,\beta)}}(\Lambda)$,

$$||v||_{\chi^{(\alpha-2,\beta-2)}} \le c|v|_{1,\chi^{(\alpha,\beta)}}. \tag{2.2}$$

Moreover, if $\alpha > -1, \beta = 0$ *or* $\alpha = 0, \beta > -1$, *then*

$$||v||_{\chi^{(\alpha,\beta)}} \le c|v|_{1,\chi^{(\alpha,\beta)}}. \tag{2.3}$$

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Lemma 2.2. (Theorem 2.2 of [1]). *For any $\phi \in \mathcal{P}_N$ and $r \geq 0$,*

$$||\phi||_{r,\chi^{(\alpha,\beta)}} \leq cN^{2r}||\phi||_{\chi^{(\alpha,\beta)}}. \tag{2.4}$$

If, in addition, $\alpha, \beta > r-1$, then

$$||\phi||_{r,\chi^{(\alpha,\beta)}} \leq cN^{r}||\phi||_{\chi^{(\alpha-r,\beta-r)}}. \tag{2.5}$$

We now turn to some orthogonal projections. For any $r \in \mathbb{N}$, let (see [1])

$$H^r_{\chi^{(\alpha,\beta)},A}(\Lambda) = \{v \mid v \text{ is measurable and } ||v||_{r,\chi^{(\alpha,\beta)},A} < \infty\}$$

where

$$||v||_{r,\chi^{(\alpha,\beta)},A} = (\sum_{k=0}^{[\frac{r-1}{2}]} ||(1-x^2)^{\frac{r}{2}-k}\partial_x^{r-k}v||^2_{\chi^{(\alpha,\beta)}} + ||v||^2_{[\frac{r}{2}],\chi^{(\alpha,\beta)}})^{\frac{1}{2}}.$$

For any real $r > 0$, the space $H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$ is defined by space interpolation. Next, for any $\mu \in \mathbb{N}$,

$$H^r_{\chi^{(\alpha,\beta)},*,\mu}(\Lambda) = \{v \mid \partial_x^\mu v \in H^{r-\mu}_{\chi^{(\alpha,\beta)},A}(\Lambda)\}, \quad H^r_{\chi^{(\alpha,\beta)},**,\mu}(\Lambda) = \{v \mid v \in H^r_{\chi^{(\alpha,\beta)},*,k}(\Lambda), 0 \leq k \leq \mu\}$$

with the following norms

$$||v||_{r,\chi^{(\alpha,\beta)},*,\mu} = ||\partial_x^\mu v||_{r-\mu,\chi^{(\alpha,\beta)},A}, \qquad ||v||_{r,\chi^{(\alpha,\beta)},**,\mu} = (\sum_{k=0}^{\mu} ||v||^2_{r,\chi^{(\alpha,\beta)},*,k})^{\frac{1}{2}}.$$

For any real $\mu > 0$, the spaces $H^r_{\chi^{(\alpha,\beta)},*,\mu}(\Lambda)$ and $H^r_{\chi^{(\alpha,\beta)},**,\mu}(\Lambda)$ are defined by space interpolation. In particular, $||v||_{r,\chi^{(\alpha,\beta)},*} = ||v||_{r,\chi^{(\alpha,\beta)},*,1}$.

Let $P_{N,\alpha,\beta} : L^2_{\chi^{(\alpha,\beta)}}(\Lambda) \to \mathcal{P}_N$ be the $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)-$orthogonal projection.

Lemma 2.3. (Theorem 2.3 of [1]). *For any $v \in H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$ and $r \geq 0$,*

$$||P_{N,\alpha,\beta}v - v||_{\chi^{(\alpha,\beta)}} \leq cN^{-r}||v||_{r,\chi^{(\alpha,\beta)},A}. \tag{2.6}$$

Lemma 2.4. (Theorem 2.4 of [1]). *If $\alpha+r > 1$ or $\beta+r > 1$, then for any $v \in H^r_{\chi^{(\alpha,\beta)},**,\mu}(\Lambda), r \geq 1$ and $0 \leq \mu \leq r$,*

$$||P_{N,\alpha,\beta}v - v||_{\mu,\chi^{(\alpha,\beta)}} \leq cN^{2\mu-r}||v||_{r,\chi^{(\alpha,\beta)},**,\mu}. \tag{2.7}$$

In particular, for any $\alpha = \beta > -1$,

$$||P_{N,\alpha,\beta}v - v||_{\mu,\chi^{(\alpha,\beta)}} \leq cN^{\sigma(\mu,r)}||v||_{r,\chi^{(\alpha,\beta)},**,\mu} \tag{2.8}$$

where $\sigma(\mu,r) = 2\mu - r - \frac{1}{2}$ for $\mu > 1$, and $\sigma(\mu,r) = \frac{3}{2}\mu - r$ for $0 \leq \mu \leq 1$.

Now let $\alpha, \beta, \gamma, \delta > -1$. We define $H^0_{\alpha,\beta,\gamma,\delta}(\Lambda) = L^2_{\chi^{(\gamma,\delta)}}(\Lambda)$, and

$$H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) = \{v \mid v \text{ is measurable and } ||v||_{1,\alpha,\beta,\gamma,\delta} < \infty\}$$

where

$$||v||_{1,\alpha,\beta,\gamma,\delta} = (|v|^2_{1,\chi^{(\alpha,\beta)}} + ||v||^2_{\chi^{(\gamma,\delta)}})^{\frac{1}{2}}.$$

For $0 < \mu < 1$, the space $H^\mu_{\alpha,\beta,\gamma,\delta}(\Lambda)$ is defined by space interpolation.

Let

$$a_{\alpha,\beta,\gamma,\delta}(u,v) = (\partial_x u, \partial_x v)_{\chi^{(\alpha,\beta)}} + (u,v)_{\chi^{(\gamma,\delta)}}.$$

The orthogonal projection $P^1_{N,\alpha,\beta,\gamma,\delta} : H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \to \mathcal{P}_N$ is a mapping such that

$$a_{\alpha,\beta,\gamma,\delta}(P^1_{N,\alpha,\beta,\gamma,\delta}v - v, \phi) = 0, \qquad \forall \phi \in \mathcal{P}_N.$$