Journal of Computational Mathematics, Vol.21, No.2, 2003, 195–200.

On the United Theory of the Family of Euler-Halley Type Methods with Cubical Convergence in Banach Spaces*[1)]

Xinghua Wang
(*Department of Mathematics, Zhejiang University, Hangzhou 310028, China*)
(*Wenzhou University, Wenzhou 325027, P. R. China*)
Chong Li
(*Department of Mathematics, Zhejiang University Huangzhou 310027, China*)
(*Academy of Mathematics and System Sciences, CAS*)

Abstract

The convergence problem of the family of Euler-Halley methods is considered under the Lipschitz condition with the L-average, and a united convergence theory with its applications is presented.

Key words: Operator equation, The family of Enler-Halley, Iterations, Cubical convergence

1. Introduction

Let E and F be real or complex Banach space with norm $\|.\|$, and let $f : D \subset E \longrightarrow F$ be a nonlinear twice differentiable operator. The family of Euler-Halley iterations with the parameter $\lambda \in [0, 2]$ for solving the operator equation $f(x) = 0$ is defined as follows:

$$x_{n+1} = T_{f,\lambda}(x_n) = x_n + u_f(x_n) + v_{f,\lambda}(x_n), \quad n = 0, 1, \cdots, \tag{1.1}$$

where

$$\begin{aligned} u_f(x) &= -f'(x)^{-1}f(x), \\ v_{f,\lambda}(x) &= -\tfrac{1}{2}f'(x)^{-1}f''(x)u_f(x)Q_{f,\lambda}(x)u_f(x), \\ Q_{f,\lambda}(x) &= \left\{I + \tfrac{\lambda}{2}f'(x)^{-1}f''(x)u_f(x)\right\}^{-1}. \end{aligned}$$

This family includes, as particular cases, the well known Euler method ($\lambda = 0$),[1, 4, 12], the Halley method ($\lambda = 1$), [3, 5, 10, 12, 18] and the convex acceleration of Newton's method or supper-Halley method ($\lambda = 2$),[6, 7, 11], so that recent interests are focused in this direction, see for example [2, 8, 9]. In particular, using a quadratic majorizing function, Argyros et al analyze the convergence of the method (1.1). However it is incorrect as shown by Han [9]. In [8], Gutierrez and Hernandez established the convergence with a cubic polynomial as the majorizing function under the classical Lipschitz condition of f'' while Han [9] established the convergence under the weak condition, so-called, γ-condition of f'', which was first presented by Wang [13, 14] when he investigated the convergence of the family of Halley methods. The purpose of the present paper is to give a united convergence theory for the family of Euler-Halley iterations such that all the known results are included as its special cases. Also some new results are obtained as the corollaries. It should be noted that this work is in sprit of Wang's idea in [15, 16].

2. Preliminaries and Lemmas

Let $D \subset E$ be a convex subset, open or closed. For $x_0 \in E, r > 0$, let $B(x_0, r)$ denote the open ball with the radius r and the center x_0 while the corresponding closed ball is denoted

* Received May 17, 2000.
[1)]This project is supported by the Special Funds for Major State Basic Research Projects(Grant No. G19990328) and the National Natural Science Foundation of China(Grant No. 10271025); also supported partly by Zhejiang Provincial Natural Science Foundation of China.

by $\overline{B(x_0,r)}$. Through the paper, we always assume that $f'(x_0)^{-1}$ exists. In order to study the convergence we require some definitions and lemmas, some of which are directly taken from [15, 16].

Definition 2.1[15]**.** A function f from D to F is called to satisfy the center Lipschitz condition in the ball $B(x_0,r)$ with the L average if

$$\|f(x)-f(x_0)\| \le \int_0^{\rho(x)} L(u)\mathrm{d}u, \quad \forall x \in B(x_0,r), \tag{2.1}$$

where $\rho(x)=\|x-x_0\|$ and L is a positive integrable function on the interval $[0,R]$ for some sufficient large number $R>0$, for example, with $\int_0^R (R-u)L(u)\mathrm{d}u = R$.

Take $r_0>0$ such that

$$\int_0^{r_0} L(u)\mathrm{d}u = 1 \tag{2.2}$$

and set

$$b=\int_0^{r_0} uL(u)\mathrm{d}u. \tag{2.3}$$

For $\beta\in(0,b]$, define

$$h(t)=\beta-t+\int_0^t (t-u)L(u)\mathrm{d}u, \quad \forall t\in[0,R]. \tag{2.4}$$

Lemma 2.1[15]**.** The function h is decreasing monotonically in $[0,r_0]$, while it is increasing monotonically in $[r_0,R]$. Moreover, if $\beta\le b$,

$$h(\beta)>0, \quad h(r_0)=\beta-b\le 0, \quad h(R)=\beta>0.$$

Consequently, h has a unique zero in each interval, respectively, which are denoted by r_1 and r_2. They satisfy

$$\beta<r_1<\frac{r_0}{b}\beta<r_0<r_2<R \tag{2.5}$$

when $\beta<b$ and $r_1=r_2$ when $\beta=b$.

Furthermore, we assume that L is a positive nondecreasing differentiable function in $[0,R]$. Then we have the following lemma.

Lemma 2.2. Let h be defined as (2.4) and $\beta\le b$. Then, for each $t\in[0,r_1]$,

(i) $H_h(t)=h'(t)^{-2}h''(t)h(t)<1$;

(ii) $T_{h,\lambda}(t)\in[0,r_1]$;

(iii) $t\le T_{h,\lambda}(t)$.

Proof. **(i)** It suffices to show that

$$g(t)=h'(t)^2-h''(t)h(t)>0.$$

Since

$$g'(t)=h'(t)h''(t)-h'''(t)h(t)=h'(t)L(t)-L'(t)h(t)\le 0,$$

so that $g(t)\ge g(r_1)=h'(r_1)^2>0$ and proves (i).

(ii) Observe that

$$T'_{h,\lambda}(t)=\frac{H_h(t)^2}{2(1-\frac{\lambda}{2}H_h(t))^2}[3(1-\frac{\lambda}{2})+\frac{\lambda}{2}(\lambda-1)H_h(t)-H_{h'}(t)].$$

Since $H_{h'}(t)$ is negative and $0<H_h(t)<1$ for each $t\in[0,r_1]$, it follows that $T'_{h,\lambda}(t)>0$ for all $t\in[0,r_1]$ and each $\lambda\in[1,2]$. Hence $T_{h,\lambda}(t)$ is monotonically increasing on $[0,r_1]$ for each $\lambda\in[1,2]$. Consequently, $T_{h,\lambda}(t)\le T_{h,\lambda}(r_1)=r_1$ for each $\lambda\in[1,2]$. On the other hand, for any $\lambda\in[0,1]$, we have

$$T_{h,\lambda}(t)\le T_{h,1}(t)\le r_1.$$

Thus (ii) holds.

(iii) This results from that $u_h(t)\ge 0$ and $v_{h,\lambda}(t)\ge 0$. ■