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THE ARTIFICIAL BOUNDARY CONDITION FOR EXTERIOR OSEEN EQUATION IN 2-D SPACE*1)

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Abstract

A finite element method for the solution of Oseen equation in exterior domain is proposed. In this method, a circular artificial boundary is introduced to make the computational domain finite. Then, the exact relation between the normal stress and the prescribed velocity field on the artificial boundary can be obtained analytically. This relation can serve as an boundary condition for the boundary value problem defined on the finite domain bounded by the artificial boundary. Numerical experiment is presented to demonstrate the performance of the method.

Key words: Artificial boundary, Exterior domain, Oseen equation.

1. Introduction

PDEs defined on exterior domain can be encountered in many application fields. Typically, the incompressible flow around a body is described by the exterior problem of Navier-Stokes equation. In numerical simulation of this kind of problems, the unboundedness of the domain is a comman difficulty. There are many methods to overcome this difficulty and the most popular one is the artificial boundary method. By introducing an artificial boundary which devides the exterior domain into a bounded part and an unbounded part, and setting up a suitable artificial boundary condition, one can reduce the original exterior problem to a boundary value problem defined only on the bounded domain which can then be solved by a suitable numerical method. The readers are referred to [3, 4, 5, 6, 7, 8, 10, 11, 12, 13] for the details of various problems.

In this paper, the analogous method is proposed for the numerical solution of steady Oseen equation. Suppose some objects are moving in $\mathcal{R}^2$ with a constant speed. By dimensionlessness procedure, the exterior Oseen equation is formulated as the following:

$$\frac{\partial \vec{u}}{\partial x} = -\nabla p + \frac{1}{\mathrm{Re}}\Delta \vec{u}, \text{ in } \Omega, \tag{1}$$

$$\nabla \cdot \vec{u} = 0, \text{ in } \Omega, \tag{2}$$

$$\vec{u} = (-1, 0), \text{ on } \Gamma, \tag{3}$$

$$\vec{u} \longrightarrow 0, \text{ when } r \longrightarrow +\infty. \tag{4}$$

where Re is the Reynolds number. Γ is the smooth boundary of the objects and Ω is the exterior domain with boundary Γ.

2. Artificial Boundary Condition

Introduce an artificial boundary $\Gamma_R \equiv \{(r,\theta)|r = R\}$ where R is large enough such that $\Gamma_R \subset \Omega$, then the artificial boundary Γ_R divides Ω into two parts: the unbounded part $\Omega_e \equiv$

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$\{(r,\theta)|r > R\}$ and the bounded part $\Omega_R \equiv \Omega\backslash\bar{\Omega}_e$. On the domain Ω_e we consider the restriction of the solution of problem (1)-(4). From [2], we know in Ω_e the velocity field $\vec{u}$ and the pressure field p has the following expression

$$u_x = \frac{\partial\phi}{\partial x} - \frac{1}{2k}\frac{\partial\chi}{\partial x} + \chi$$
$$u_y = \frac{\partial\phi}{\partial y} - \frac{1}{2k}\frac{\partial\chi}{\partial y}$$
$$p = -\frac{\partial\phi}{\partial x}$$

where $k = \frac{\mathrm{Re}}{2}$, ϕ and χ are two multi-valued functions satisfying the following equations

$$\Delta\phi = 0, \text{ in } \Omega_e$$
$$(\Delta - 2k\frac{\partial}{\partial x})\chi = 0, \text{ in } \Omega_e.$$

Furthermore, they have the following expansions (see [2])

$$\phi = \frac{a_0}{2}\log r - \frac{b_0}{2}\theta - \sum_{n=1}^{+\infty}\frac{1}{n}\left(\frac{R}{r}\right)^n (a_n\cos n\theta + b_n \sin n\theta)$$
$$\chi = \frac{c_0}{2K_0(kR)}e^{kx}K_0(kr) + \frac{d_0}{K_0(kR)}\int_0^{-\infty} e^{k(x+\xi)}\frac{\partial}{\partial y}K_0(kr_\xi)d\xi$$
$$+e^{kx}\sum_{n=1}^{+\infty}\frac{K_n(kr)}{K_n(kR)}(c_n\cos n\theta + d_n\sin n\theta)$$

where $r_\xi = \sqrt{(x-\xi)^2 + y^2}$. Here and hereafter, K_n and I_n denote the first kind and second kind modified Bessel functions respectively (see [1] for detail). After a computation, on Γ_R we have

$$u_r = \sum_{m=0}^{+\infty}\delta_m\left(\sum_{n=0}^{+\infty}\Phi^1_{mn}c_n + \frac{a_m}{R}\right)\cos m\theta + \sum_{m=1}^{+\infty}\left(\sum_{n=0}^{+\infty}\Phi^2_{mn}d_n + \frac{b_m}{R}\right)\sin m\theta$$
$$u_\theta = \sum_{m=0}^{+\infty}\delta_m\left(\sum_{n=0}^{+\infty}\Psi^2_{mn}d_n - \frac{b_m}{R}\right)\cos m\theta + \sum_{m=1}^{+\infty}\left(\sum_{n=0}^{+\infty}\Psi^1_{mn}c_n + \frac{a_m}{R}\right)\sin m\theta$$

where

$$\Phi^1_{mn} = \frac{1}{2}\delta_n\left(I'_{m+n} + I'_{m-n} - \frac{K'_n}{K_n}(I_{m+n} + I_{m-n})\right), m \geq 0, n \geq 0$$
$$\Phi^2_{mn} = \frac{1}{2}(I'_{m-n} - I'_{m+n} - \frac{K'_n}{K_n}(I_{m-n} - I_{m+n})), m \geq 0, n > 0$$
$$\Phi^2_{mn} = \frac{I_{m-1} - I_{m+1}}{2}, m \geq 0, n = 0$$
$$\Psi^1_{mn} = \frac{1}{2kR}\delta_n((2n-m)I_{m-n} - (2n+m)I_{m+n}), m \geq 0, n \geq 0$$
$$\Psi^2_{mn} = -\frac{1}{2kR}((2n-m)I_{m-n} + (2n+m)I_{m+n}), m \geq 0, n > 0$$
$$\Psi^2_{mn} = I'_m + \frac{K'_0}{K_0}I_m, m \geq 0, n = 0$$