Journal of Computational Mathematics, Vol.20, No.5, 2002, 533–542.

A NEW SMOOTHING APPROXIMATION METHOD FOR SOLVING BOX CONSTRAINED VARIATIONAL INEQUALITIES*

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Abstract

In this paper, we first give a smoothing approximation function of nonsmooth system based on box constrained variational inequalities and then present a new smoothing approximation algorithm. Under suitable conditions,we show that the method is globally and superlinearly convergent. A few numerical results are also reported in the paper.

Key words: Box constrained variational inequalities, Smoothing approximation, Global convergence, Superlinear convergence.

1. Introduction

Consider the following box constrained variational inequalities of finding an $x \in X$, such that

$$f(x)^T(y-x) \geq 0, \quad \forall y \in X \tag{1.1}$$

where f is a smooth mapping from R^n into itself, set X has the following box form:

$$X = \{x \in R^n | a \leq x \leq b\} \tag{1.2}$$

with $a = (a_1, a_2, \cdots, a_n)^T,\ b = (b_1, b_2, \cdots, b_n)^T$ and $-\infty \leq a_i < b_i \leq +\infty, i = 1, \cdots, n$.

It is easily shown that problem (1.1) is equivalent to

$$H(x) = 0 \tag{1.3}$$

where $H(x) = (H_1(x), \cdots, H_n(x))^T,\ H_i(x)(i = 1, \cdots, n)$ defined by

$$H_i(x) = \begin{cases} x_i - a_i, & x_i - f_i(x) \leq a_i, \\ f_i(x), & a_i < x_i - f_i(x) < b_i, \\ x_i - b_i, & x_i - f_i(x) \geq b_i, \end{cases} \tag{1.4}$$

Generally, mapping H is nonsmooth. Qi and Chan(see[6]) established a so-called successive approximation method, which changes (1.3) into an equivalent nonsmooth equations of the following form:

$$H(x) \equiv f_k(x) + g_k(x) = 0 \tag{1.5}$$

where f_k and g_k are mappings from R^n into itself with f_k being F-differentiable and g_k not, but its norm relatively small. Global convergence of this method is obtained and numerical examples are given to illustrate its usefulness.

Another approach for solving problem (1.3) is the smoothing methods(see[7, 8, 9]). The feature of smoothing methods is to construct a smoothing approximation function $H(\cdot, \cdot):\ R^n \times R_{++} \to R^n$ of H such that for any $\epsilon > 0,\ H(\cdot, \epsilon)$ is continuously differentiable and

$$\|H(x) - H(x, \epsilon)\| \to 0\ as\ \epsilon \to 0^+\ for\ all\ x \in R^n$$

* Received June 23, 1999; Final revised November 30, 2000.

and then to find a solution of (1.3) by solving the following problem for a given sequence $\{\epsilon_k\}, k = 0, 1, \cdots,$

$$H(x, \epsilon_k) = 0. \tag{1.6}$$

In [7], Chen and Mangasarian introduced a class of smoothing approximation functions for nonlinear complementarity problems. And Gabriel and Moré extended Chen-Mangasarian's smoothing approximation functions to box constrained variational inequalities(see[9]).

In this paper, we will present a new smoothing approximation method for solving problem (1.1). Under mild conditions, we prove that the sequence $\{x_k\}$ generated by this method is bounded and each accumulation point is a solution of problem (1.3).

This paper is organized as follows. In the next section, we define the Jacobian consistency property and construct a new smoothing approximation function of H. And then we describe the algorithm in detail. In section 3, we show that the algorithm is globally and superlinearly convergent. Finally, some numerical examples are given in section 4.

We let $\|\cdot\|$ denote the Euclidean norm of R^n and let

$$R_+ = \{\epsilon | \epsilon \geq 0,\ \epsilon \in R\}, \quad R_{++} = \{\epsilon | \epsilon > 0,\ \epsilon \in R\},$$

we denote the set of all nonnegative integers by N=$\{0,\ 1,\ 2, \cdots\}$.

2. The Algorithm

Let $F : R^n \to R^m$ be locally Lipschitz continuous. According to Rademacher's theorem, H is differentiable almost everywhere. Let D_F be the set where F is differentiable. The B-derivative of F is defined by(see[3])

$$\partial_B F(x) = \{\lim_{x_k(\in D_F) \to x} F'(x_k)\}.$$

The generalized Jacobian of F at x in the sense of Clarke(see[1]) is

$$\partial F = conv \partial_B F(x).$$

In this paper, for the function F, we use a kind of generalized Jacobian, denote by $\partial_C F$ and defined as(see[10])

$$\partial F = \partial F_1(x) \times \partial F_2(x) \times \cdots \partial F_n(x).$$

Next we give the definition of the Jacobian consistency property.

Definition 2.1. Let $H(\cdot)$ be a Lipschitz function in R^n. We call $H(\cdot, \epsilon) : R^n \times R_{++} \to R^n$ a smoothing approximation function of $H(\cdot)$ if $H(\cdot, \epsilon)$ is continuously differentiable with respect to variable x and there exists a constant $c > 0$ such that for any $x \in R^n$ and $\epsilon \in R_{++}$,

$$\|H(x, \epsilon) - H(x)\| \leq c\epsilon \tag{2.1}$$

Further, if for any $x \in R^n$,

$$\lim_{\epsilon \downarrow 0} dist(H'_x(x, \epsilon), \partial_C H(x)) = 0 \tag{2.2}$$

then we say $H(\cdot, \epsilon)$ satisfies the Jacobian consistency property.

We now construct a new smoothing approxiomation function $H(x, \epsilon) = (H_i(x, \epsilon))$ of $H(x)$ defined by (1.4) as follows

$$H_i(x, \epsilon) = \begin{cases} x_i - a_i - \frac{(x_i - f_i(x) - a_i + \epsilon)^2}{4\epsilon}, & i \in \alpha(x), \\ x_i - b_i + \frac{(x_i - f_i(x) - b_i - \epsilon)^2}{4\epsilon}, & i \in \beta(x), \\ H_i(x), & otherwise \end{cases} \tag{2.3}$$

where $\alpha(x) = \{i : |x_i - f_i(x) - a_i| < \epsilon\}$, $\beta(x) = \{i : |x_i - f_i(x) - b_i| < \epsilon\}$. It isn't difficult to find that $H(x, \epsilon)$ is continuously differential while $\epsilon \leq \min\limits_{1 \leq i \leq n} \{\frac{1}{2}(b_i - a_i)\}$. And Jacobian $H'_x(x, \epsilon)$ is