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RATE OF CONVERGENCE OF SCHWARZ ALTERNATING METHOD FOR TIME-DEPENDENT CONVECTION-DIFFUSION PROBLEM*1)

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Abstract

This paper provides a theoretical justification to a overlapping domain decomposition method applied to the solution of time-dependent convection-diffusion problems. The method is based on the partial upwind finite element scheme and the discrete strong maximum principle for steady problem. An error estimate in $L^\infty(0,T;L^\infty(\Omega))$ is obtained and the fact that convergence factor $\rho(\tau,h)\to 0$ exponentially as $\tau,h\to 0$ is also proved under some usual conditions.

Key words: Rate of convergence, Schwarz alternating method, Convection-diffusion problem.

1. Introduction

Schwarz alternating procedure is the earliest method of domain decomposition approach in the context of partial differential equations. It has been paid attention by mathematician and engineer since 1980's although it was proposed in 1869 by H.A.Schwarz. Via maximum priciple it is easy to prove that there exists a convergence factor $\rho\in(0,1)$ such that the error reduce with geometric rate: $\|e^{k+1}\|_\infty\le\rho^k\|e^0\|_\infty$. It is easy to understand that the convergence factor ρ depends on the size of the overlapping domain, but that fact had not been proved untill to the middle of 1980's. For Laplace equations in rectangular domain, with Fourier seriers and Schwarz alternating procedure Evans, Shao, Kang, Chen and Tang[4][11][9] found that the convergence factor depend exponentially on size of overlapping. For second order elliptic partial defferential equations in domain $\Omega\subset\mathbf{R}^d(d\ge 2)$, P.-L.Lions proved that[8][9]: Set Ω has been decomposed two overlapping subdomain $\Omega=\Omega_1\cup\Omega_2$, $\Omega_1\cap\Omega_2=\Omega_{12}\neq\emptyset$. Let $\gamma_1=\partial\Omega_1\cap\Omega_2$, $\gamma_2=\partial\Omega_2\cap\Omega_1$ and

$$\delta=\mathrm{dist}(\gamma_1,\gamma_2)>0$$

denote the degree of the overlapping. Then there exists a constant $\mu>0$ dependent on the coefficients of equation and subdomain Ω_1 such that the convergence factor of Schwarz alternating procedure $\rho=\exp(-\mu\delta^2)$, i.e. the convergence factor depends exponentially on the degree of overlapping.

Domain decomposition method constitutes an important actively developed approach to approximate realization of implicit schemes for unsteady problems. However, to the author's knowledge, the discussion of convergence factor for discrete approximation of time-dependent

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problems is rare. For one-dimensional diffusion equation with constant coefficients Rui H.-X. proved [10] the convergence factor ρ also depends exponentially on the degree of overlapping and $\rho(\tau, h) \to 0$ exponentially as $\tau, h \to 0$, but the method of proof depended on analytic solution of one-dimensional diffusion problem with constant coefficients and it is not easy to analyse more general equations in the same way.

In this paper, we discuss a domain decomposition method with overlapping applied to the solution of convection-diffusion problems. For the sake of simplicity, we consider two-dimensional problem $(\mathbf{P})$ as follows

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nabla \cdot (a\nabla u) + \mathbf{b} \cdot \nabla u + cu = f, & (x,t) \in \Omega \times (0,T]; \\ u(x,0) = u_0(x), & x \in \overline{\Omega}; \\ u(x,t) = g(x,t), & (x,t) \in \partial\Omega \times [0,T]. \end{cases}$$

where $\Omega \subset \mathbf{R}^2$ with boundary $\partial\Omega$. The coefficents a, $\mathbf{b}$, c and the functions f, u_0, g are smooth enough. Moreover $a(x) \ge a_0 \ge 0$, $c(x) \ge 0$. For simplicity we set $g(x,t) = 0$.

In the next section we shall describe a kind of finite element scheme to approach problem $(\mathbf{P})$ and in the third section we present two kind of Schwarz alternating procedure to solve the numerical approximation of problem $(\mathbf{P})$. In the last section we study the rate of convergence for the above Schwarz alternating procedure and then an error estimate in $L^\infty(0,T;L^\infty(\Omega))$ is obtained, the fact that convergence factor $\rho(\tau,h) \to 0$ exponentially as $\tau, h \to 0$ is also proved under some usual conditions.

2. Partial upwind finite element scheme

In this section we shall describe a kind of finite element scheme to approach problem $(\mathbf{P})$. Special attention is paid particularly to problems with convection dominating over diffusion (i.e. $|\mathbf{b}| >> a$) because of difficulties with an accurate resolution of the so-call boundary layers. For discretization of problem $(\mathbf{P})$, efficient finite difference or finite element schemes are usually based on the use of upwinding or artificial viscosity. But the full upwind scheme, as well as the artificial viscosity scheme, involve additional viscosity, which may cause excessive dullness of numerical solutions. We use an efficient scheme so-called partial upwind finite element in order to recover the shape of the exact solution as sharply as possible.

We denote a scalar product in $L^2(\Omega)$ by $(\cdot,\cdot)$ and define bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ in space $H^1(\Omega)$ as follows

$$a(u,v) = (a\nabla u, \nabla v), \quad b(u,v) = (\mathbf{b} \cdot \nabla u, v).$$

Then the weak form of problem $(\mathbf{P})$ will be that for $t \in (0,T]$ to find $u(t) \in H_0^1(\Omega)$ such that $u(0) = u_0$ and

$$(u'(t), v) + a(u,v) + b(u,v) + (cu, v) = (f,v), \ \forall v \in H_0^1(\Omega) \tag{2.1}$$

For discretization of above problem $(\mathbf{P})$, we use an efficient scheme called partial upwind finite element [3][5]. Let Ω_h be a polygonal approximation of the domain Ω. Consider a regular triangulation[2] $\mathcal{T}_h = \{e\}$ defined over $\bar{\Omega}_h$, where each element e of $\mathcal{T}_h$ is a closed triangle. We denote the internal vertexes by x_i with $i = 1, \cdots, N_p$; the boundary vertexes on $\partial\Omega$ by x_i with $i = N_p + 1, \cdots, M_p$. We put h_e to be the maximum side length and κ_e to be the minimum perpendicular length of triangle e. Set

$$h = \max\{h_e;\ e \in \mathcal{T}_h\}, \quad \kappa = \min\{\kappa_e;\ e \in \mathcal{T}_h\}.$$

We assume that the regular triangulation $\mathcal{T}_h$ is quasi-uniform, i.e. there exist positive constants c_1, c_2 such that

$$c_1 \le \frac{h_e}{h} \le 1 < \frac{h_e}{\kappa_e} \le c_2, \quad \forall e \in \mathcal{T}_h.$$