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BIVARIATE FRACTAL INTERPOLATION FUNCTIONS ON RECTANGULAR DOMAINS*1)

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Abstract

Non-tensor product bivariate fractal interpolation functions defined on gridded rectangular domains are constructed. Linear spaces consisting of these functions are introduced. The relevant Lagrange interpolation problem is discussed. A negative result about the existence of affine fractal interpolation functions defined on such domains is obtained.

Key words: Fractal, Bivariate functions, Interpolation.

1. Introduction

In 1986 Barnsley[2] constructed a sort of continuous functions by using certain *Iterated Function Systems* (IFS). Such a function f is defined on a compact interval, which is partitioned into a number of subintervals, and is said to be self-affine since the restriction of f within one of the subintervals is just a composition of a scaling and a translation of f plus an affine function. The graph of f, which interpolates a set of given points, has usually a non-integral fractal dimension and is then called a *Fractal Interpolation Function*, abbreviated FIF. FIF serves a useful tool for constructing, modelling, simulating, and approximating functions which display some sort of self-similarity under magnification and find its applications in several areas such as image compression and wavelet analysis (cf. [1, 3, 4, 5, 8]).

There are two natural ways to extend the idea of FIF to the case of two variables. Geromino *et al.*[6] and Massopust[7] deal with continuous functions with the property of self-affinity defined on the triangulated triangular domains, whose graphs are so called *fractal surfaces*. Unfortunately, the gridded rectangular domain, i.e., the rectangular domain divided into a number of quadrangles, especially rectangles which is most used in the applications of Computer Graphics, is hardly considered. Massopust[8] suggests a construction by trivial taking the tensor product of two univariate FIFs. The drawback of this tensor product scheme is explicit: the derived function is uniquely determined by the its evaluation along a pair of adjacent sides of the rectangular domain, thus it cannot be used to fit in with a set of data more extensively sampled.

In this paper we shall develop the idea of the space of fractal functions introduced by Qian[9] to study the bivariate fractal functions defined on rectangular domains. Such functions are continuous, but not tensor products of univariate FIFs. They can be designated to interpolating given data distributed over the grid points. Their graphs, as those of univariate FIFs, can be generated by IFSs of certain simple forms. We shall show that they are distinct in various aspects from their analogies defined on intervals and triangular domains. For example, it will be proved that the "affine" fractal functions defined on a gridded rectangular domain does, in essence, not exist. For this reason one should be satisfied with the bivariate FIFs generated by IFSs of other forms as simple as possible.

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This paper is organized as follows. In Section 2 we introduce bivariate FIFs defined on rectangular domains . In Section 3 we discuss the bivariate FIFs generated by affine IFSs and prove that they are usually affine functions. In Section 4 we construct a class of spaces of bivariate FIFs suitable for interpolation and study the structures of such spaces.

2. Linear Spaces of Bivariate Fractal Interpolation Functions

Let $m,n \geq 2$ be two integers . Denote $M = \{1,\cdots,m\}$ and $N = \{1,\cdots,n\}$. Let $-\infty < x_0 < x_1 < \cdots < x_m < \infty$ and $-\infty < y_0 < y_1 < \cdots < y_n < \infty$. Denote by D the rectangular region $[x_0,x_m]\times[y_0,y_n]$ and $D_{i,j} = [x_{i-1},x_i]\times[y_{j-1},y_j]$ for all $i \in M$ and $j \in N$. Denoted by Δ the partition of D given by $D = \bigcup_{i,j} D_{i,j}$. Moreover, let Λ denote a fixed matrix $(s_{i,j})_{m\times n}$ with $-1 < s_{i,j} < 1$. For all $i \in M$ and $j \in N$, define maps $A_i : [x_0,x_m] \longrightarrow [x_{i-1},x_i]$ and $B_j : [y_0,y_n] \longrightarrow [y_{j-1},y_j]$ by

$$A_i(x) = \frac{x_i - x_{i-1}}{x_m - x_0}(x - x_0) + x_{i-1}, \qquad x \in [x_0,x_m]$$

and

$$B_j(y) = \frac{y_j - y_{j-1}}{y_n - y_o}(y - y_o) + y_{j-1}, \qquad y \in [y_0,y_n]$$

respectively.

We denote by $\mathrm{C}(D)$ the linear space of all real-valued continuous functions defined on D and by $\mathrm{Lip}(D)$ the set of all bivariate Lipschitzian functions defined on D. Obviously $\mathrm{Lip}(D)$ is a linear subspace of $\mathrm{C}(D)$. Given a family of functions $\phi_{i,j} \in \mathrm{Lip}(D)$, $i \in M$, $j \in N$, define mappings $T_{i,j} : D\times\mathbf{R} \longrightarrow D\times\mathbf{R}$ by

$$T_{i,j}(x,y,z) = (A_i(x), B_j(y), s_{i,j}\,z + \phi_{i,j}(x,y)),$$

for $(x,y,z) \in D\times\mathbf{R}$, $i \in M$, $j \in N$. We now obtain an IFS

$$\{D\times\mathbf{R}\,;\; T_{i,j} :\; i \in M,\; j \in N\}. \tag{2.1}$$

The following is easy to be proved.

Lemma 2.1. *The IFS (2.1) has a unique invariant set.*

Definition 2.1. *An IFS of the form (2.1) is said to be generating if its unique invariant set G is the graph of some $f \in \mathrm{C}(D)$, in which case we also write*

$$\{D\times\mathbf{R}\,;\; T_{i,j} :\; i \in M,\; j \in N\} \to f \tag{2.3}$$

and say that f is generated by the IFS.

One can easily check the following by Definition 2.1.

Proposition 2.1. *The IFS (2.1) is generating if and only if there exists some $f \in \mathrm{C}(D)$ such that*

$$f(A_i(x), B_j(y)) = s_{i,j} f(x,y) + \phi_{i,j}(x,y), \quad (x,y) \in D\,, i \in M,\; j \in N, \tag{2.3}$$

in which case the relation (2.2) holds.

Theorem 2.1. *IFS (2.1) is generating if and only if there exist functions $p_0, p_1 \in \mathrm{C}[x_0,x_m]$ and $q_0, q_1 \in \mathrm{C}[y_0,y_n]$ such that*

$$\begin{cases} p_0(x_0) = q_0(y_0), & p_0(x_m) = q_1(y_0), \\ p_1(x_0) = q_0(y_n), & p_1(x_m) = q_1(y_n), \end{cases} \tag{2.4}$$

and

$$\phi_{i,j+1}(x,y_0) - \phi_{i,j}(x,y_n) = s_{i,j}p_1(x) - s_{i,j+1}p_0(x), \qquad i \in M,\; 1 \leq j \leq n-1, \tag{2.5}$$

$$\phi_{i+1,j}(x_0,y) - \phi_{i,j}(x_m,y) = s_{i,j}q_1(y) - s_{i+1,j}q_0(y), \qquad 1 \leq i \leq m-1,\; j \in N, \tag{2.6}$$

$$p_0(A_i(x)) = s_{i,1}p_0(x) + \phi_{i,1}(x,y_0), \;\; i \in M, \tag{2.7}$$