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ON THE CONVERGENCE OF PROJECTOR-SPLINES FOR THE NUMERICAL EVALUATION OF CERTAIN TWO-DIMENSIONAL CPV INTEGRALS[*1)]

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Abstract

In this paper, product formulas based on projector-splines for the numerical evaluation of 2-D CPV integrals are proposed. Convergence results are proved, numerical examples and comparisons are given.

Key words: 2-D Cauchy principal value integral, Tensor product, projector-splines.

1. Introduction

We consider the numerical evaluation of Cauchy principal value integrals of the form

$$J(f;z,\vartheta)=\int_a^b\int_{\tilde a}^{\tilde b} w_1(x)w_2(\tilde x)\frac{f(x,\tilde x)}{(x-z)(\tilde x-\vartheta)}\,\mathrm{d}x\,\mathrm{d}\tilde x \tag{1.1}$$

where $z\in(a,b)$, $\vartheta\in(\tilde a,\tilde b)$, the weight functions $w_1(x)$, $w_2(\tilde x)$ and the function f are such that $J(f;z,\vartheta)$ exists.

The numerical evaluation of the integrals (1.1) are of two types: global and local. The global methods have generally to be used when f is differentiable with 'small' derivatives. However, one of the difficulties wich occur in the use of global methods usually based on orthogonal polynomials, lies in the fact that a greater accuracy in approximating (1.1) requires to increase the number of the nodes coinciding with the zeros of above polynomials. Therefore, when the weight functions w_1, w_2 are different from the classical Jacobi weights, the evaluation of the nodes requires a considerable computational effort.

Besides, global methods are generally not appropriate when f behave 'badly' in some subinterval of $[a,b]\times[\tilde a\times\tilde b]$, then for such integrals a local method with no restriction on the choice of the nodes would have to be preferred.

In this paper we will consider an approximation function of the form:

$$Q_{N\tilde N}f(x,\tilde x)=\sum_{i=1-k}^{N-1}\sum_{\tilde\imath=1-k}^{\tilde N-1}(\lambda_{i\tilde\imath}\tilde\lambda_{i\tilde\imath}f)B_{i\tilde\imath k}(x,\tilde x) \tag{1.2}$$

in which the operators $\lambda_{i\tilde\imath}$, $\tilde\lambda_{i\tilde\imath}$ are such that $Q_{N\tilde N}$ is the tensor product of two one-dimensional projector-splines and we will examine a cubature rule for (1.1), considering that it can be written in the form

$$\begin{aligned}J(f;z,\vartheta) &= \int_a^b\int_{\tilde a}^{\tilde b} w_1(x)w_2(\tilde x)\frac{f(x,\tilde x)-f(z,\vartheta)}{(x-z)(\tilde x-\vartheta)}\,\mathrm{d}x\,\mathrm{d}\tilde x+\\ &+ f(z,\vartheta)\int_a^b\frac{w_1(x)}{x-z}\,\mathrm{d}x\int_{\tilde a}^{\tilde b}\frac{w_2(\tilde x)}{\tilde x-\vartheta}\,\mathrm{d}\tilde x,\end{aligned} \tag{1.3}$$

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and then, it can be approximated by

$$\begin{aligned} J_{N\tilde{N}}(f;z,\vartheta) &= \int_a^b\!\!\int_{\tilde{a}}^{\tilde{b}} w_1(x)w_2(\tilde{x})\frac{Q_{N\tilde{N}}f(x,\tilde{x})-Q_{N\tilde{N}}f(z,\vartheta)}{(x-z)(\tilde{x}-\vartheta)}\,\mathrm{d}x\,\mathrm{d}\tilde{x}\,+ \\ &+ \; f(z,\vartheta)\int_a^b \frac{w_1(x)}{x-z}\,\mathrm{d}x\int_{\tilde{a}}^{\tilde{b}}\frac{w_2(\tilde{x})}{\tilde{x}-\vartheta}\,\mathrm{d}\tilde{x}. \end{aligned} \tag{1.4}$$

This paper is organized as follows. In Section 2 we will present some preliminaries and summarize numerical thecniques to be used; in Section 3 we will prove the convergence of the integration rules here proposed and we give conditions for their uniform convergence for (ζ,ϑ) belonging to any closed interval contained in $(a,b)\times(\tilde{a},\tilde{b})$. Finally, in Section 4, some numerical results are presented and compared with those obtained by using the method proposed in [2].

2. Preliminaries

Given $\Omega := [a,b]\times[\tilde{a},\tilde{b}]$, let $\{Y_n\}$ and $\{\tilde{Y}_{\tilde{n}}\}$ be two sequences of partitions of $I := [a,b]$ and $\tilde{I} := [\tilde{a},\tilde{b}]$ respectively:

$$Y_n := \{a = y_{0n} < y_{1n} < ... < y_{nn} = b\},\quad \tilde{Y}_{\tilde{n}} := \{\tilde{a} = \tilde{y}_{0\tilde{n}} < \tilde{y}_{1\tilde{n}} < ... < \tilde{y}_{\tilde{n}\tilde{n}} = \tilde{b}\}.$$

If $h_i = y_{i+1} - y_i$ and $\tilde{h}_{\tilde{\imath}} = \tilde{y}_{\tilde{\imath}+1} - \tilde{y}_{\tilde{\imath}}$, we define

$$\delta_1 = \min_{1\le i\le n} h_{i-1},\quad \delta_2 = \min_{1\le \tilde{\imath}\le \tilde{n}} \tilde{h}_{\tilde{\imath}-1}. \tag{2.1}$$

Let $\overline{\Delta}_1$, $\overline{\Delta}_2$ be the norms of the partitions Y_n and $\tilde{Y}_{\tilde{n}}$ respectively, given by

$$\overline{\Delta}_1 = \max_{1\le i\le n} h_{i-1},\quad \overline{\Delta}_2 = \max_{1\le \tilde{\imath}\le \tilde{n}} \tilde{h}_{\tilde{\imath}-1}. \tag{2.2}$$

We say that the collection of partitions $\{Y_n\times\tilde{Y}_{\tilde{n}} : n = n_1, n_2...; \tilde{n} = \tilde{n}_1,\tilde{n}_2,...\}$ of Ω, is quasi-uniform $(q.u.)$ if there exists a positive constant A such that

$$\frac{\overline{\Delta}_i}{\delta_j} \le A,\quad 1\le i,j\le 2 \tag{2.3}$$

and we assume that

$$\overline{\Delta}_1 \to 0 \quad \text{as} \quad n\to\infty,\quad \overline{\Delta}_2 \to 0 \quad \text{as} \quad \tilde{n}\to\infty. \tag{2.4}$$

Let $\{d_{in}\}_1^{n-1}$, $\{\tilde{d}_{\tilde{\imath}\tilde{n}}\}_1^{\tilde{n}-1}$ be two sequences of positive integers with $d_{in} \le k-1$, $\tilde{d}_{\tilde{\imath}\tilde{n}} \le \tilde{k}-1$, where $k, \tilde{k}$ are assigned integers greater than 1, and let π be the non-decreasing sequence $\{x_i\}_0^N$ obtained from Y_n by repeating y_{in} exactly d_i times (thus $N = \sum_i^{n-1} d_i + 1$); similarly, let $\tilde{\pi}$ be the non-decreasing sequence $\{\tilde{x}_{\tilde{\imath}}\}_0^{\tilde{N}}$ obtained from $\tilde{Y}_{\tilde{n}}$ (thus $\tilde{N} = \sum_{\tilde{\imath}}^{\tilde{n}-1}\tilde{d}_{\tilde{\imath}} + 1$). We denote with $S_{\pi k}$ and $\tilde{S}_{\tilde{\pi}\tilde{k}}$ the polynomial spline spaces of order k and $\tilde{k}$ respectively. We shall call a sequence of spline spaces $\{S_{\pi k}\times\tilde{S}_{\tilde{\pi}\tilde{k}}\}$ $q.u.$ if they are based on a sequence of $q.u.$ partitions.

We can suppose, without loss of generality, $k = \tilde{k}$.

It is well known that considering the extended partitions $\pi_e = \{x_i\}_{i=1-k}^{N+k-1}$ and $\widetilde{\pi}_e = \{\widetilde{x}_i\}_{\widetilde{i}=1-k}^{\widetilde{N}+k-1}$, the normalized B-splines $\{B_{ik}(x)\}_{i=1-k}^{N-1}$ and $\{\tilde{B}_{\tilde{\imath}k}(\tilde{x})\}_{\tilde{\imath}=1-k}^{\tilde{N}-1}$ constitue a basis compactly supported for $S_{\pi k}$ and $\tilde{S}_{\tilde{\pi}k}$ respectively. By the above univariate normalized B-splines we may generate a collection of bivariate B-splines, defined on $[x_{1-k}, x_{N+k-1}]\times[\tilde{x}_{1-k}, \tilde{x}_{\tilde{N}+k-1}]$,

$$B_{i\tilde{\imath}k}(x,\tilde{x}) = B_{ik}(x)\tilde{B}_{\tilde{\imath}k}(\tilde{x}).$$