Journal of Computational Mathematics, Vol.20, No.1, 2002, 79–88.

ERROR ANALYSIS AND GLOBAL SUPERCONVERGENCES FOR THE SIGNORINI PROBLEM WITH LAGRANGE MULTIPLIER METHODS*[1)]

Ping Luo
(*Department of Computer Science, Tsinghua University, Beijing 100084, China*)

Guo-ping Liang
(*Institute of Mathematics, Academy of Mathematics and System, Chinese Academy of Sciences, Beijing 100080, China*)

Abstract

In this paper, the finite element approximation of the Signorini problem is studied by using Lagrange multiplier methods with piecewise constant elements. Optimal error bounds are obtained and iterative algorithm and its convergence are given. Furthermore, global superconvergences are proved.

Key words: Lagrange multipliers, Elastic contact, Error estimates, Superconvergences.

1. Introduction

It is well known that contact problems have always occupied a position of special importance in the mechanics of solids. So it attracted particular attention of engineers and computational experts. In the last ten years, with the development of the theory of variational inequalities, a lot of new results and methods on contact problems have been reported. We refer to Brezzi, Hager and Raviart [3,4], Falk [8], Glowinski, Lions and Trémoliérés [10], Haslinger [11], Haslinger and Hlaváček [12,13], Kikuchi and Oden [15] for details and survey in this field.

Superconvergence estimates for the finite element methods are well studied in many papers. We refer to Křížek and Neittanmäki [16], Lin and Xu [20], Lin and Zhu [21,32], Křížek [17] and Wahlbin [30] for more details.

We shall discuss in this paper the classical example of the role of variational inequalities in the formulation of contact problems in solid mechanics – the so-called Signorini problem describing the contact of a linearly elastic body with a rigid frictionless foundation and show that the finite element approximation of the Lagrange multipliers methods with piecewise constant element is of order one in accuracy with the energy norm. Furthermore, we find that the error estimates with the energy norm are half a order higher than the usual optimal error bounds if the partition of Ω is almost a uniform piecewise strongly regular mesh and the solution is smooth enough.

We consider the deformation of a body unilaterally supported by a frictionless rigid foundation and subjected to body forces f and surface tractions t applied to a portion Γ_F of the body's surface Γ. The body is fixed along a portion Γ_D of its boundary and we denote by Γ_c a portion of body which is a candidate contact surface. Let $\mathbf{u} = (u_1, u_2)$ and $\sigma = (\sigma_{ij}), 1 \le i, j \le 2$, denote arbitrary displacement and stress fields in the body respectively. If we take on the specific form $\sigma_{ij}(\mathbf{u}) = E_{ijkl} u_{k,l}(x)$, where E_{ijkl} are the components of Hooke's tensor which may depend on x and have the symmetry properties

$$E_{ijkl} = E_{jikl} = E_{klij},\ 1 \le i, j, k, l \le 2.$$

* Received October 18, 1999; Final revised March 6, 2001.
1) This research was supported by NSFC grant number 19971050 and by 973 grant number G1998030420.

Then the Signorini problem consists in finding the displacement field $\mathbf{u}$ such that

$$\begin{aligned} -(E_{ijkl}u_{k,l})_{,j} &= f_i, \text{ in } \Omega, \\ u_i &= 0, \text{ on } \Gamma_D, \\ E_{ijkl}u_{k,l}n_j &= t_i, \text{ on } \Gamma_F, \\ \left.\begin{aligned} \sigma_{T_i}(\mathbf{u}) &= 0, \ \sigma_n(\mathbf{u})(u_n - g) = 0, \\ u_n - g &\leq 0, \ \sigma_n(\mathbf{u}) \leq 0, \end{aligned}\right\} &\text{on } \Gamma_c, \end{aligned} \tag{1}$$

where $u_{k,l} = \dfrac{\partial u_k}{\partial x_l}$, $\mathbf{n}$ and g denote the outward unit normal vector on Γ_c and the normalized initial gap, respectively, and

$$\begin{aligned} u_n &= \mathbf{u}\cdot\mathbf{n} = u_i n_i, \ \sigma_n(\mathbf{u}) = \sigma_{ij}(\mathbf{u})n_i n_j, \\ \sigma_{T_i}(\mathbf{u}) &= \sigma_{ij}(\mathbf{u})n_j - \sigma_n(\mathbf{u})n_i, \ 1 \leq i,j,k,l \leq 2, \end{aligned}$$

here $\sigma_{T_i}(\mathbf{u}), \sigma_n(\mathbf{u})$ denote the tangential and the normal components of the stress vector, respectively, and the usual summation convention on repeated indices is used.

Let Ω be a bounded domain with a sufficiently smooth boundary. We will use the usual Sobolev space $W^{m,p}(\Omega)$ consisting of real valued functions defined on Ω with derivatives up to order m in $L^p(\Omega)$ and the norm on $W^{m,p}(\Omega)$ is denoted by $||\cdot||_{m,p,\Omega}$. In particular, we define

$$H^m(\Omega) = W^{m,2}(\Omega) \quad \text{and} \quad ||\cdot||_{m,\Omega} = ||\cdot||_{m,2,\Omega}.$$

Let $(\cdot,\cdot)$ denote the L^2-inner product and

$$\Sigma = int(\Gamma - \Gamma_D), \ \Gamma_F \subset \Sigma, \ \Gamma_F \cap \Gamma_c = \phi, \ \overline{\Gamma}_c \subset \Sigma,$$

where int denotes interior of a set. For vector-valued function $\mathbf{v}$ with components v_i in $H^m(\Omega)$, we shall use the following notations

$$\begin{aligned} \mathbf{v} \in \mathbf{H^m}(\Omega) &= \{(v_1,v_2)|v_i \in H^m(\Omega), 1 \leq i \leq 2\}, \\ ||\mathbf{v}||_{m,\Omega} &= \{\sum_{i=1}^{2}\int_\Omega \sum_{|\alpha|\leq m} |D^\alpha v_i(x)|^2 dx\}^{\frac{1}{2}}, \\ \mathbf{V} &= \{\mathbf{v} \in \mathbf{H^1}(\Omega) |\, \gamma_D(\mathbf{v}) = 0, \text{ in } H^{\frac{1}{2}}(\Gamma_D)\}, \\ \mathbf{K} &= \{\mathbf{v} \in \mathbf{V} |\, \gamma^0_{\Sigma_n}(\mathbf{v}) - g \leq 0, \quad \text{a.e. on } \Gamma_c\}, \\ \gamma_D &: \mathbf{H}^1(\Omega) \to H^{\frac{1}{2}}(\Gamma_D), \gamma_D \text{ being the trace map from } \mathbf{H}^1(\Omega), \\ \gamma^0_{\Sigma_n} &: \mathbf{V} \to H^{\frac{1}{2}}(\Sigma), \gamma^0_{\Sigma_n} \text{ being the normal trace map from } \mathbf{V}. \end{aligned}$$

We see that the set $\mathbf{K}$ is a nonempty closed convex subset of $\mathbf{V}$ and the normal trace operator $\gamma^0_{\Sigma_n}$ is a surjective linear continuous map. Let $H^{-\frac{1}{2}}(\Gamma_c)$ denote the dual space of $H^{\frac{1}{2}}(\Gamma_c)$ and its norm be defined by

$$||v||_{-\frac{1}{2},\Gamma_c} = \sup_{0\neq\varphi\in H^{\frac{1}{2}}(\Gamma_c)} \frac{|\int_{\Gamma_c} v\varphi|}{||\varphi||_{\frac{1}{2},\Gamma_c}},$$

where

$$||v||^2_{\frac{1}{2},\Gamma_c} = ||v||^2_{0,\Gamma_c} + |v|^2_{\frac{1}{2},\Gamma_c}, \quad |v|^2_{\frac{1}{2},\Gamma_c} = \int_{\Gamma_c}\int_{\Gamma_c} \frac{(v(x)-v(y))}{(x-y)^2} dxdy.$$