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PROBABILISTIC NUMERICAL APPROACH FOR PDE AND ITS APPLICATION IN THE VALUATION OF EUROPEAN OPTIONS*

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Abstract

This paper suggests a probabilistic numerical approach for a class of PDE. First of all, by simulating Brownian motion and using Monte-Carlo method, we obtain a probabilistic numerical solution for the PDE. Then, we prove that the probabilistic numerical solution converges in probability to its solution. At the end of this paper, as an application, we give a probabilistic numerical approach for the valuation of European Options, where we see volatility σ, interest rate r and divident rate D_0 as functions of stock S, respectively.

Key words: Brownian motion, Probabilistic numerical solution, European options.

1. Introduction

This paper is aimed to give a probabilistic numerical approach for PDE. Probabilistic numerical method can get the solution one by one, which differs from other numerical methods, such as the finite element and finite difference method, and realize total parallel computing easily. Another advantage of this method is that it suits for problems of high-dimension because it is dimension-independent.

Consider the following Cauchy problem of convection-diffusion equations. For simplicity, we will only discuss 1-dimension problem. The method can be easily extended to higher dimensional problems. Find $u = u(x,t)$ such that

$$(I)\begin{cases} u_t = a(x)u_{xx} + b(x)u_x + c(x)u & (1.1)\\ u(x,0) = \phi(x) & (1.2)\end{cases}$$

where $u(x,t)$ is an unknown function defined on $R^1 \times (0,T], \phi(x)$ is an initial function which satisfies some conditions. $a(x) > 0, b(x)$ and $c(x)$ are given functions.

In the first place, let $u(x,t) = v(y,t)$, $y = y(x)$. Then, we have

$$\begin{aligned} u_t &= v_t \\ u_x &= v_y \cdot y_x \\ u_{xx} &= v_{yy} \cdot (y_x)^2 + v_y \cdot y_{xx} \end{aligned}$$

and

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$$v_t = a(x)(y_x)^2 v_{yy} + (a(x)y_{xx} + b(x)y_x)v_y + c(x)v. \tag{1.3}$$

Furthermore, let $a(x)(y_x)^2 = \frac{1}{2}$, namely $y(x) = \int_0^x \frac{1}{\sqrt{2a(z)}}dz$. Note that $y(x)$ is a strictly increasing function. There exists a unique inverse function $x = x^{-1}(y)$. Replace all the x in (1.3) by $x^{-1}(y)$, we can get

$$(I')\begin{cases} v_t = \frac{1}{2}v_{yy} + B(y)v_y + C(y)v & (1.4) \\ v(y,0) = \Phi(y), & (1.5) \end{cases}$$

where

$$\begin{aligned} B(y) &= a(x^{-1}(y))y_{xx}(x^{-1}(y)) + b(x^{-1}(y))y_x(x^{-1}(y)) \\ C(y) &= c(x^{-1}(y)) \\ \Phi(y) &= \phi(x^{-1}(y)). \end{aligned}$$

From above transformations we see that we only need to give a probabilistic numerical solution for problem (I').

Now we introduce some notations and symbols. Let $\{\xi_t, \Im_t, t \geq 0\}$ be a 1-dim standard Brownian motion(for short, BM) starting at y. P_y is a probabilistic measure about BM and E_y is the expectation related to P_y. Define

$$\begin{aligned} W_s &\equiv \int_0^s B(\xi_r)d\xi_r - \frac{1}{2}\int_0^s |B(\xi_r)|^2 dr \\ L_s &\equiv \int_0^s C(\xi_r)dr \\ Z_s &\equiv W_s + L_s \qquad s \geq 0. \end{aligned}$$

2. Probabilistic Numerical Approach for (I')

2.1. Simulating the Path of Brownian Motion

Suppose that $\eta_i^0, \eta_i^1, \cdots, \eta_i^N (i \geq 1)$ are independent random variables with uniform distribution on $(0,1)$. Let

$$\widehat{\xi}_i^k = \sqrt{-2\ln \eta_i^{k-1}}\cos 2\pi\eta_i^k \qquad i \geq 1, k = 1, 2, \cdots, N$$

then $\widehat{\xi}_i^1, \cdots, \widehat{\xi}_i^N$ $(i \geq 0)$ are independent random variables with normal distribution $N(0,1)$

Let $y \in R^1$ and divide time interval $[0,T]$ $(0 < T < \infty)$ into:

$$0 = t_0 < t_1 < \cdots < t_m < \cdots < t_n = T, \quad t_i - t_{i-1} = h.$$

Take

$$\begin{aligned} &\xi_i^k - \xi_{i-1}^k = \sqrt{h}\widehat{\xi}_i^k \quad i \geq 0, k = 1, 2, \cdots, N \\ &\xi_0^k \equiv y. \end{aligned}$$