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THE CONVERGENCE ON A FAMILY OF ITERATIONS WITH CUBIC ORDER*1)

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Abstract

In this paper, we establish a convergent theorem for a family of iterations with cubic order by using general convergence hypotheses. A sharp error estimate is also given explicitly.

Key words: Convergence, Majorizing method, γ-Condition, Error estimate.

1. Introduction

Let E and F be real or complex Banach spaces and $f : D \subseteq E \to F$ be a nonlinear twice differentiable operator. For solving the equation

$$f(x) = 0, \tag{1.1}$$

consider a one-parametered family of iterations

$$x_{n+1} = x_n - \left[I + \frac{1}{2} H_f(x_n)(I - \lambda H_f(x_n))^{-1} \right] f'(x_n)^{-1} f(x_n), \; n = 0, 1, \cdots, \tag{1.2}$$

where $H_f(x) = f'(x)^{-1} f''(x) f'(x)^{-1} f(x)$ and $0 \le \lambda \le 1$. This family is cubically convergent (see [3, 7]) and includes, as particular cases, Chebyshev method ($\lambda = 0$, see [5, 9]), Halley method ($\lambda = \frac{1}{2}$, see [1, 4, 6, 10]) and super-Halley method ($\lambda = 1$, see [7]).

In [3], I.K. Argyros et al analyze the convergence of (1.2) by using the quadratic majorizing function. But we found that there are some mistakes in the analysis. Especially, when the norm of the operator is estimated by the majorizing function, the second term of the last expression on page 270 in [3] is wrong . The reason for this is that the second term in the identity (4) in [3] is not always positive for all λ. In addition, the authors of [3] also give a convergence condition, but is not uniform for the parameter λ. In fact, just as we analyze below, for the iteration (1.2), it is unable to give a uniform condition independent on the parameter λ by the quadratic majorizing function. Therefore, other majorizing functions are required. In [7], M.A. Hernandez et al use cubic polynomial as a majorizing function and establish a convergence theorem. But the disadvantage of using the cubic majorizing function is that for the iteration (1.2),the error estimate can't be represented by an explicit form. Besides, the convergence condition is also too complicated.

To overcome these disadvantages above, we present more general or even weaker convergence hypotheses and establish a convergence theorem with a very simple convergence condition. Meanwhile, a sharp error estimate is given explicitly.

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2. Majorizing Function

Let β, γ be positive numbers and $h : [0, (1 - \frac{1}{\sqrt{2}})\frac{1}{\gamma}] \to R$ be a real function defined by

$$h(t) = \beta - t + \frac{\gamma\, t^2}{1 - \gamma\, t}\ . \tag{2.1}$$

Write $\alpha = \beta \cdot \gamma$. Then when $\alpha \leq 3 - 2\sqrt{2}$, h has two real roots

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = \frac{1 + \alpha \mp \sqrt{(1+\alpha)^2 - 8\alpha}}{4\gamma}.$$

They satisfy the inequality

$$\beta \leq r_1 \leq (1 + \frac{1}{\sqrt{2}})\beta \leq (1 - \frac{1}{\sqrt{2}})\frac{1}{\gamma} \leq r_2 \leq \frac{1}{2\gamma}. \tag{2.2}$$

For $x_0 \in D$, suppose $f'(x_0)^{-1}$ exists. First, we give the following γ-condition ([9])

Definition 1. *Let h be defined by* (2.1). *Then h is said to satisfy the γ-condition about f at x_0 if*

$$||f'(x_0)^{-1} f(x_0)|| \leq \beta, \quad ||f'(x_0)^{-1} f''(x_0)|| \leq 2\gamma, \tag{2.3}$$

$$||f'(x_0)^{-1} f'''(x)|| \leq \frac{6\gamma^2}{(1 - \gamma||x - x_0||)^3}, \ \ \forall\, x \in D,\ \ ||x - x_0|| < (1 - \frac{1}{\sqrt{2}})\frac{1}{\gamma}. \tag{2.4}$$

If h satisfies the γ-condition about f, then h is also said to be a majorizing function of f.

Lemma 1. *Assume h satisfies the γ-condition about f at x_0. If $||x - x_0|| < (1 - \frac{1}{\sqrt{2}})\frac{1}{\gamma}$, then we have*
(a) $||f'(x_0)^{-1} f''(x)|| \leq h''(||x - x_0||)$;
(b) $f'(x)^{-1}$ *exists and*

$$||f'(x)^{-1} f'(x_0)|| \leq -\frac{1}{h'(||x - x_0||)}.$$

Proof. By the definition, we have

$$\begin{aligned} ||f'(x_0)^{-1} f''(x)|| &\leq ||f'(x_0)^{-1} f''(x_0)|| + ||\int_0^1 f'(x_0)^{-1} f'''(x_0 + s(x - x_0))|| \cdot ||x - x_0|| ds \\ &\leq h''(0) + \int_0^1 h'''(s||x - x_0||)||x - x_0|| ds = h''(||x - x_0||). \end{aligned}$$

Hence (a) follows.
On the other hand, by Taylor formula,

$$f'(x_0)^{-1} f'(x) = I + f'(x_0)^{-1} f''(x_0)(x - x_0) + \int_0^1 f'(x_0)^{-1} f'''(x_0 + s(x - x_0))(1 - s)(x - x_0)^2 ds.$$

Thus, by (2.3), (2.4) and the nonpositivity of h' , we have

$$\begin{aligned} ||f'(x_0)^{-1} f'(x) - I|| \quad &\leq \quad \int_0^1 ||f'(x_0)^{-1} f'''(x_0 + s(x - x_0))|| \cdot ||x - x_0||^2 (1 - s) ds \\ &\quad\ + ||f'(x_0)^{-1} f''(x_0)|| \cdot ||x - x_0|| \\ &\leq \quad \int_0^1 h'''(s||x - x_0||)||x - x_0||^2 (1 - s) ds + h''(0)||x - x_0|| \\ &\leq \quad h'(||x - x_0||) - h'(0) = 1 + h'(||x - x_0||) < 1 \end{aligned} \tag{2.5}$$